Math 185-5 Fall 2015, Homework 1 Solutions to Graded Problems

5.7 Observe that

$$\begin{aligned}\frac{|P(z)|}{|a_n||z^n|} &= \frac{|a_0 + a_1 z + \ldots + a_n z^n|}{|a_n||z^n|}\\ &\leq \frac{|a_0| + |a_1||z| + \ldots |a_n||z^n|}{|a_n||z^n|} \quad \text{(triangle inequality)}\\ &= \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + 1.\\ &< \frac{1}{n} + \frac{1}{n} + \ldots + 1 < 2,\\ &\quad \text{when } |z| > n \cdot \max\{|a_0|, \ldots, |a_{n-1}|, 1\}\end{aligned}$$

9.1 (a)

$$arg(z) = Arg(-2) - Arg(1 + \sqrt{3}i) + 2\pi n = -\pi - (\pi/3) + 2\pi n = -4\pi/3 + 2\pi n$$

The principal value is thus $-4\pi/3 + 2\pi = 2\pi/3$ (by taking $n = 1$).

(b) $Arg(z) = \arctan(-1/\sqrt{3}) = -\pi/6$, so $arg(z) = 6 \cdot Arg(z) + 2\pi n = -\pi + 2\pi n$ and $Arg(z^6) = \pi$ (corresponding to $n = 1$).

11.3 We begin by writing $z = -8 - 8\sqrt{3}i$ in polar form. The magnitude is $|z| = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = \sqrt{4^4} = 16$, and the argument is the set of all θ satisfying

$$\cos\theta = -8/16 = -1/2 \quad \sin\theta = -8\sqrt{3}/16 = -\sqrt{3}/2.$$

Thus, we must have $arg(z) = \{-2\pi/3 + 2\pi n : n \in \mathbb{Z}\}$, $Arg(z) = -2\pi/3$, and $z = 16e^{-i2\pi/3}$.

A fourth root is any solution to the equation $w^4 = z$. If $w = re^{i\phi}$, then

$$(re^{i\phi})^4 = r^4 e^{i4\phi} = 16e^{-i2\pi/3}.$$

Since two complex numbers are equal iff their magnitudes and arguments are equal, we must have

$$r^4 = 6 \Rightarrow r = 2,$$

(there is just one root since we insist that $r > 0$) and for some $k \in \mathbb{Z}$:

$$4\phi = -2\pi/3 + 2\pi n \Rightarrow \phi = -\pi/6 + (\pi/2)n.$$

We need only consider $n = 0, \ldots, 3$ since after this the roots start repeating. Thus, the roots are

$$\begin{aligned}2e^{-i\pi/6} &= 2(\cos(-\pi/6) + i\sin(-\pi/6)) = \sqrt{3} - i, \text{(principal)}\\ 2e^{-i\pi/6 + i\pi/2} &= 1 + \sqrt{3}i\\ 2e^{-i\pi/6 + i\pi} &= -\sqrt{3} + i\\ 2e^{-i\pi/6 + i3\pi/2} &= -1 - \sqrt{3}i.\end{aligned}$$

12.10 Assume by way of contradiction that z_0 is an accumulation point, and choose $\rho = \min\{|z_0 - z_k| : k = 1, \ldots n\}$, where the minimum exists because there are finitely many points. Then the punctured neighborhood $D^o(z_0, \rho) = \{z : 0 < |z - z_0| < \rho\}$ contains no points of S, so z_0 cannot be an accumulation point.

#2 Assume $z\overline{w} \neq 1$.

$$
\begin{aligned}
&\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \\
&\iff |w-z| < |1-\overline{w}z| \\
&\iff |w-z|^2 < |1-\overline{w}z|^2 \\
&\iff (w-z)(\overline{w-z}) < (1-\overline{w}z)(\overline{1-\overline{w}z}) \\
&\iff (w-z)(\overline{w}-\overline{z}) < (1-\overline{w}z)(1-w\overline{z}) \\
&\iff w\overline{w} - w\overline{z} - z\overline{w} + z\overline{z} < 1 - w\overline{z} - z\overline{w} + w\overline{w}z\overline{z} \\
&\iff |w|^2 + |z|^2 < 1 + |w|^2|z|^2 \quad \text{cancelling } w\overline{z} + z\overline{w} = 2\mathrm{Re}(w\overline{z}) \\
&\iff |w|^2|z|^2 - |w|^2 - |z|^2 + 1 > 0 \\
&\iff (|w|^2 - 1)(|z|^2 - 1) > 0.
\end{aligned}
$$

The last inequality is satisfied whenever $|w| < 1$ and $|z| < 1$. If either is equal to one it (and everything above it) becomes an equality.

Fixing w with $|w| < 1$, we conclude that $|F(z)| < 1$ when $|z| < 1$ and $|F(z)| = 1$ when $|z| = 1$. It is easy to check that $F(w) = 0$ and $F(0) = w$.