Math 185-5 Fall 2015, Homework 1 Solutions to Graded Problems

5.7 Observe that

\[
\frac{|P(z)|}{|a_n||z^n|} = \frac{|a_0 + a_1z + \ldots + a_nz^n|}{|a_n||z^n|} \\
\leq \frac{|a_0| + |a_1||z| + \ldots + |a_n||z^n|}{|a_n||z^n|} \quad \text{(triangle inequality)} \\
= \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + 1. \\
< \frac{1}{n} + \frac{1}{n^2} + \ldots + 1 < 2, \\
\text{when } |z| > n \cdot \max\{|a_0|, \ldots, |a_{n-1}|, 1\}
\]

9.1 (a)

\[\arg(z) = \Arg(-2) - \Arg(1 + \sqrt{3}i) + 2\pi n = -\pi - (\pi/3) + 2\pi n = -4\pi/3 + 2\pi n\]

The principal value is thus \(-4\pi/3 + 2\pi = 2\pi/3\) (by taking \(n = 1\)).

(b) \(\arg(z) = \arctan(-1/\sqrt{3}) = -\pi/6\), so \(\arg(z) = 6 \cdot \arg(z) + 2\pi n = -\pi + 2\pi n\) and \(\arg(z^6) = \pi\) (corresponding to \(n = 1\)).

11.3 We begin by writing \(z = -8 - 8\sqrt{3}i\) in polar form. The magnitude is \(|z| = \sqrt{(-8)^2 + (-8\sqrt{3})^2} = \sqrt{4^2} = 16\), and the argument is the set of all \(\theta\) satisfying

\[\cos \theta = -8/16 = -1/2 \quad \sin \theta = -8\sqrt{3}/16 = -\sqrt{3}/2.\]

Thus, we must have \(\arg(z) = \{-2\pi/3 + 2\pi n : n \in \mathbb{Z}\}, \arg(z) = -2\pi/3,\) and \(z = 16e^{-i2\pi/3}.\)

A fourth root is any solution to the equation \(w^4 = z\). If \(w = re^{i\phi}\), then

\[(re^{i\phi})^4 = r^4e^{4i\phi} = 16e^{-i2\pi/3}.\]

Since two complex numbers are equal iff their magnitudes and arguments are equal, we must have

\[r^4 = 6 \Rightarrow r = 2,\]

(there is just one root since we insist that \(r > 0\)) and for some \(k \in \mathbb{Z}:\)

\[4\phi = -2\pi/3 + 2\pi n \Rightarrow \phi = -\pi/6 + (\pi/2)n.\]

We need only consider \(n = 0, \ldots, 3\) since after this the roots start repeating. Thus, the roots are

\[2e^{-i\pi/6} = 2(\cos(-\pi/6) + i\sin(-\pi/6)) = \sqrt{3} - i,\] (principal)

\[2e^{-i\pi/6 + i\pi/2} = 1 + \sqrt{3}i,\]

\[2e^{-i\pi/6 + i\pi} = -\sqrt{3} + i,\]

\[2e^{-i\pi/6 + i3\pi/2} = -1 - \sqrt{3}i.\]
12.10 Assume by way of contradiction that \(z_0\) is an accumulation point, and choose \(\rho = \min\{|z_0 - z_k| : k = 1, \ldots, n\}\), where the minimum exists because there are finitely many points. Then the punctured neighborhood \(D^\rho(z_0, \rho) = \{z : 0 < |z - z_0| < \rho\}\) contains no points of \(S\), so \(z_0\) cannot be an accumulation point.

#2 Assume \(z\bar{w} \neq 1\).

\[
\left| \frac{w - z}{1 - \bar{w}z} \right| < 1
\]

\(\iff\)

\[
|w - z| < |1 - \bar{w}z|
\]

\(\iff\)

\[
|w - z|^2 < |1 - \bar{w}z|^2
\]

\(\iff\)

\[
(w - z)(\bar{w} - \bar{z}) < (1 - \bar{w}z)(1 - \bar{w}z)
\]

\(\iff\)

\[
(w - z)(\bar{w} - \bar{z}) < (1 - \bar{w}z)(1 - \bar{w}z)
\]

\(\iff\)

\[
w\bar{w} - w\bar{z} - z\bar{w} + z\bar{z} < 1 - w\bar{z} - z\bar{w} + w\bar{w}z\bar{z}
\]

\(\iff\)

\[
|w|^2 + |z|^2 < 1 + |w|^2|z|^2 \quad \text{cancelling } w\bar{z} + z\bar{w} = 2\Re(w\bar{z})
\]

\(\iff\)

\[
|w|^2|z|^2 - |w|^2 - |z|^2 + 1 > 0
\]

\(\iff\)

\[
(|w|^2 - 1)(|z|^2 - 1) > 0.
\]

The last inequality is satisfied whenever \(|w| < 1\) and \(|z| < 1\). If either is equal to one it (and everything above it) becomes an equality.

Fixing \(w\) with \(|w| < 1\), we conclude that \(|F(z)| < 1\) when \(|z| < 1\) and \(|F(z)| = 1\) when \(|z| = 1\). It is easy to check that \(F(w) = 0\) and \(F(0) = w\).