Math 185-5 Fall 2015, Homework 14

Optional. Due December 9 at the review session

This homework assignment is optional, but doing it will help you understand the proof of the prime number theorem and will serve as good practice for the final exam.

1. Prove that
$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \ldots = \sum_{n:2\nmid n} \frac{1}{n^s},$$
whenever $\mathrm{Re}(s) > 1$, where $2 \nmid n$ means that 2 does not divide n.

   Using the unique factorization property, generalize this to show that
$$\prod_{\text{prime } p<N} \left(1 - \frac{1}{p^s}\right)\zeta(s) = \sum_{n:p\nmid n, \forall p<N} \frac{1}{n^s}.$$
   Using the convergence of $\zeta(s)$ for $\mathrm{Re}(s) > 1$, conclude that
$$\lim_{N\to\infty} \prod_{\text{prime } p<N} \left(1 - \frac{1}{p^s}\right)\zeta(s) = 1,$$
   establishing the Euler product formula.

2. (a) Show that the functions
$$f_N(s) = \prod_{\text{prime } p<N} \left(1 - \frac{1}{p^s}\right).$$
   converge uniformly to $\zeta(s)$ in every strip $\{z : \mathrm{Re}(s) \geq a\}$ where $a > 1$ (hint: use the uniform convergence of $\zeta(s)$).

   (b) Using the argument principle and the fact that $f_N(s) \neq 0$ when $\mathrm{Re}(s) > 1$, conclude that $\zeta(s)$ has no zeros in $\mathrm{Re}(s) > 1$ (hint: to apply the principle, assume for contradition that f has a zero z_0. Since this zero is isolated there must be a circle around z_0 on which $|f(s)| \geq \delta$ for some δ. Now apply uniform convergence to deduce that $|f_N(s)| \geq \delta/2$ on this circle, then establish that
$$\oint_C \frac{f_N'(s)}{f_N(s)} ds \to \oint_C \frac{f'(s)}{f(s)} ds,$$
   and finally apply the argument principle.)

   This is a special case of a more general theorem called *Hurwitz's theorem* .

3. Show that
$$\frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{y^s}{s} ds = \begin{cases} 0 & y > 1 \\ 1 & y < 1 \end{cases},$$
whenever $y > 0$ and $c > 0$. (hint: close the contour with a box to the right of the vertical line for one of the cases, and with a box to the left for the other case).

Recall that this integral allowed us to extract the partial sum

$$\sum_{n<x} \Lambda(n)$$

from the infinite series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$