Math 185-5 Fall 2015, Homework 13 Selected Solutions

1*
- Let $S = f(D(z_0, r))$ be the image of $D(z_0, r)$. Since f is 1-1 in $D(z_0, r)$, there is an inveres function $g : S \to D(z_0, r)$ such that $g(f(z)) = z$ for all $z \in D(z_0, r)$. Since $w_0 \in S$, we know by the open mapping theorem that there is a neighborhood $D(w_0, \rho) \subset S$; the restriction $g : D(w_0, \rho) \to D(z_0, r)$ is the required function.
- Observe first that since f is continuous at z_0, its inverse g must be continuous at w_0. To see why, note that every sequence $w_n \to w_0$ is the image of a sequence $\{z_n\}$ under the bijection $w_n = f(z_n)$; moreover $z_n \to z_0$ as $n \to \infty$, since if we had $z_n \to z_0'$ for some $z_0' \neq z_0$ this would imply $w_n = f(z_n) \to f(z_0') \neq w_0$ by continuity and local 1-1ness of f. Thus, we have $w_n \to w_0$ iff $z_n \to z_0$ for every sequence $w_n = f(z_n)$, provided $z_n \in D(z_0, r)$.

  Applying this to the difference quotient for $g'(z_0)$ and appealing to the first part of the question, we find:

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} = \lim_{z \to z_0} \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} = \frac{1}{f'(z_0)},$$

  which exists since $f'(z_0) \neq 0$.

2* (a) Since u is harmonic we know that $u_{xx} + u_{yy} = 0$, and both u_x and u_y are continuous, which implies that $u_{xy} = u_{yx}$. These properties imply that u_x and $-u_y$ satisfy the Cauchy-Riemann equations:

$$(u_x)_x = (-u_y)_y \qquad (u_x)_y = -(-u_y)_x,$$

so again by continuity, we conclude that the function $g(x + iy) = u_x(x, y) - iu_y(x, y)$ is analytic at every (x, y) in the domain. In particular, u_x has a harmonic conjugate, namely $-u_y$, there.

(b) If D is simply connected then by Cauchy-Goursat the function $g(z)$ defined above has an antiderivative in D, say $g(z) = f'(z)$ for some $f = U + iV$. Taking a derivative, we see that $U_x - iU_y = u_x - iu_y$, which after equating real and imaginary parts yields $U_x = u_x$ and $U_y = u_y$. Integrating with respect to x and y, we conclude that $U(x, y) = u(x, y) + C_1(y)$ and $U(x, y) = u(x, y) + C_2(x)$ for some functions C_1, C_2, which is only possible if $U = u + C$ for some complex constant C. But now $f - C = u + iV$ is analytic in D, so V is the required harmonic conjugate.

3* Suppose u is nonconstant in a domain D, and let $(x_0, y_0) \in D$. Let $D' \subset D$ be an open disk centered at (x_0, y_0); since D' is simply connected u must have a harmonic conjugate in D' and we can write $u = \text{Re}(f(z))$ for some analytic f. By the open mapping theorem, the image $f(D')$ is open, in particular it contains an open disk centered at $f(z_0)$ for $z_0 = x_0 + iy_0$. Thus, $u(D') = \text{Re}(f(D'))$ contains an open interval centered at $u(x_0, y_0)$. This interval must contain a point of magnitude greater than $|u(x_0, y_0)|$, so (x_0, y_0) cannot be a local maximum.

4* To check that u is harmonic we simply verify that it satisfies Laplace's equation (omitted).

If u is harmonic in $\mathbb{C} \setminus \{0\}$ then $\frac{1}{2}u$ is also harmonic there, so $\frac{1}{2}u = \text{Re}(f(z))$ for some f analytic in $\mathbb{C} \setminus \{0\}$. Observe that $\frac{1}{2}u(x, y) = \ln(\sqrt{x^2 + y^2}) = \text{Re}(\text{Log}(x + iy))$ where Log is the principal branch of the logarithm. Since both f and $\text{Log}(z)$ are analytic in the slit plane $\mathbb{C} \setminus (-\infty, 0]$ and $\text{Re}(f(z) - \text{Log}(z)) = 0$ there, the identity theorem tells us that $f(z) = \text{Log}(z) + C$ in $\mathbb{C} \setminus (-\infty, 0]$, for some constant C. Thus, we have $f'(z) = \frac{d}{dz}\text{Log}(z) = 1/z$ in $\mathbb{C} \setminus (-\infty, 0]$.

Repeating exactly the same argument with another branch $\text{Log}_0(z)$ with branch cut equal to the nonnegative real axis $[0,\infty)$, we find that $f'(z) = 1/z$ in $\mathbb{C}\setminus[0,\infty)$ (since all branches of $\log(z)$ have the same derivative). Altogether, we have $f'(z) = 1/z$ in $\mathbb{C}\setminus\{0\}$. But the existence of an antiderivative implies that

$$\oint_{|z|=1} \frac{1}{z}dz = 0,$$

which we know is false since it is equal to $2\pi i$ (the fundamental integral). Thus, u cannot have a harmonic conjugate in $\mathbb{C}\setminus\{0\}$.

114.6* (a) Every inverse function of $w = z^2$ has the form

$$w = \rho e^{i\phi} \mapsto \sqrt{\rho}e^{i\phi/2}$$

where $\phi = \arg(w)$ is some branch of the argument. In this case we would like to find a branch with two properties: (i) it is analytic at w_0, noting that $\arg(w_0) = 0 + 2\pi n$. In particular, this means the branch cut should not pass through the positive real axis. (ii) it maps w_0 to $z_0 = 2$, which requires $\phi_0/2 = 0 + 2\pi n'$.

Both these properties are satisfied if we take $\phi = \arg(w) \in (-\pi,\pi)$, the principal branch, so the desired inverse is just $\sqrt{\rho}e^{i\phi/2}$ with $\phi \in (-\pi,\pi)$.

(b) In this case $z_0 = -2$ so we would like a branch of ϕ analytic at ϕ_0 but with $\phi_0/2 = -\pi + 2\pi n'$, since $z_0 = -2$. This is achieved by taking $\phi \in (\pi, 3\pi)$, which has a branch cut on the negative real axis and maps $\phi_0 = 2\pi$ to π. The function is thus

$$\sqrt{\rho}e^{i\phi/2} \quad \phi \in (\pi, 3\pi).$$

(c) Here $z_0 = -i$ and $w_0 = z_0^2 = -1$, so $\phi_0 = \arg(w_0) = \pi + 2\pi n$ and we would like a branch analytic on the negative real axis, such that $\phi_0/2 = \arg(-i) = -\pi/2 + 2\pi n'$. These requirements are met by taking $\phi \in (-\pi/2, 3\pi/2)$.

I find the book's approach somewhat awkward. An alternate way to think about this is in terms of branches of $\log$ *rather than* $\arg$. *In particular, a square root is just a function of type* $\exp(\frac{1}{2}\log(w))$ *for some branch of* $\log$. *The image of such a branch is a strip of vertical width* 2π, *so the image of* $\frac{1}{2}\log(w)$ *is a half-strip of width* π. *One way to picture what we are doing is that we are just choosing a half-strip in a way that its exponential contains the original point* z_0.

6
- By a Taylor expansion write
$$f(z) = f(z_0) + (z - z_0)^k g(z)$$
for some $g(z_0) \neq 0$ in a neighborhood of z_0.
- Since $g(z_0) \neq 0$ the function
$$h(z) = \exp(\frac{1}{k}\text{Log}(g(z))$$
is analytic and nonzero in a neighborhood of z_0, for some appropriate branch of the logarithm, and $h(z)^k = g(z)$ there. Thus,
$$f(z) - f(z_0) = ((z - z_0)h(z))^k = \phi(z)^k$$
where $\phi(z) = (z - z_0)h(z)$, in a neighborhood N of z_0.
- Observe that $\phi(z_0) = 0$ and $\phi'(z_0) = (z-z_0)'h(z_0) + 0\cdot h'(z_0) = h(z_0) \neq 0$. By the inverse function theorem there is a function ϕ^{-1} defined on a neighborhood N of 0 such that $\phi^{-1}(\phi(z)) = z$ for all $z \in \phi^{-1}(N)$. Let N^o denote the punctured disk $N \setminus \{0\}$. Observe that the kth power map $p_k : N^o \to \mathbb{C}$ given by $p_k(w) = w^k$ is k-to-1, since for any $w \in N^o$ we have $w^k = (e^{2\pi i/k}w)^k = \ldots = (e^{2\pi i(k-1)/k}w)^k$, all k of these numbers are contained in N^o since they have the same magnitude,

and there are no other $w' \in N^o$ with $w'^k = w^k$. Since $\phi : \phi^{-1}(N^o) \to N^o$ is a bijection, the composition $f = p_k \circ \phi : \phi^{-1}(N^o) \to \mathbb{C}$ is also k-to-1.

Note that the set $\phi^{-1}(N^o)$ is an open set, but is not necessarily a punctured disk. We can, if we like, restrict to a punctured disk $D^o \subset \phi^{-1}(N^o)$ set but then f will be at most k-to-1 rather than exactly k-to-1 in this disk, i.e., every point in the image will have at most k preimages rather than exactly k. However, since $f(D^o)$ contains a punctured disk $N'^o \subset N^o$, it will not be k'-to-1 for any $k' < k$. I should have made this point clearer in the question: either I should have asked for a punctured open set centered at z_0 in which the function is exactly k-to-1, or for a punctured disk in which it is at most k-to-1 but not at most k'-to-1 for any $k' < k$.

7* Our strategy for solving this problem consists of three steps: (1) find a Mobius transformation mapping the unit disk to the upper half plane. (2) solve the problem in the upper half plane using an appropriate linear combination of $\mathrm{Arg}(z - z_i)$ functions. (3) Compose these to get a solution in the disk.

Since the boundary conditions are discontinuous at the points $z_0 = e^{i0} = 1, z_1 = e^{i\pi} = -1, z_2 = e^{i3\pi/2} = -i$, it will be most convenient to find a transformation which maps these points (as opposed to some other three points) to $0, 1\infty$. The required transformation is:

$$w = T(z) = \frac{z-1}{z+i}\frac{-1+i}{-1-1}.$$

We may check that this maps the interior of $D(0,1)$ to the *upper* half plane either by plugging in

$$T(0) = \frac{-1}{i}\frac{-1+i}{-2} = \frac{-i+i^2}{-2} = \frac{1+i}{2},$$

which indeed lies in the upper half plane, or by appealing to the fact that T is conformal at every point except $z_2 = i$, so in particular it preserves the fact that zero lies to the left of the curve as one moves from $z_0 = 1$ to $z_1 = -1$.

The three arcs corresponding to $\theta \in (0,\pi), \theta \in (\pi, 3\pi/2), \theta \in (3\pi/2, 2\pi)$ get mapped to the intervals $(0,1), (1,\infty), (-\infty, 0)$, respectively. Thus, the corresponding Dirichlet problem in the upper half plane is: find a function harmonic in $\{\mathrm{Im}(w) > 0\}$ with boundary values:

$$\begin{cases} -1 & (-\infty, 0) \\ 0 & (0,1) \\ 1 & (1,\infty) \end{cases}.$$

As in the lecture notes, we consider the family of functions:

$$h(w) = A_0 \cdot \mathrm{Arg}(w-0) + A_1 \cdot \mathrm{Arg}(w-1) + B,$$

where Arg is chosen to have a branch cut on the negative imaginary axis, so that $\mathrm{Arg}(w) = \mathrm{Im}(\mathrm{Log}(w))$ for the corresponding branch of $\mathrm{Log}(w)$, analytic in the upper halfplane. Again, the reason it is sufficient to consider such functions is that $\mathrm{Arg}(w - w_0)$ is a step function with a jump at w_0, and we can realize any boundary values by looking at a combination of step functions.

We solve for A_0, A_1, B by evaluating at points in the three intervals:

$$-1 = h(-1) = A_0 \cdot (\pi) + A_1 \cdot (\pi) + B,$$

$$0 = h(1/2) = A_0 \cdot (0) + A_1 \cdot (\pi) + B,$$

$$1 = h(2) = A_0 \cdot (0) + A_1 \cdot (0) + B.$$

Working in reverse order, we find that:

$$B = 1 \qquad A_1 = -B/\pi = -1/\pi \qquad A_0 = -1/\pi - A_1 - B/\pi = -1 + 1/\pi - 1/\pi = -1/\pi.$$

Thus, the function $h(w) = 1 - \frac{1}{\pi}\text{Arg}(w) - \frac{1}{\pi}\text{Arg}(w-1) = \text{Im}(f(w))$ for $f(w) = i - \pi^{-1}\text{Log}(w) - \pi^{-1}\text{Log}(w)$ solves the problem in the upper half plane, and the composition

$$H(z) = \text{Im}(f(T(z)) = 1 - \frac{1}{\pi}\text{Arg}(\frac{z-1}{z+i}\frac{1-i}{2}) - \frac{1}{\pi}\text{Arg}(\frac{z-1}{z+i}\frac{1-i}{2} - 1),$$

is a solution in the disk.

I did not explicitly ask you to do this, but since harmonic functions are functions from $\mathbb{R}^2$ to $\mathbb{R}$, it is more appropriate (though sometimes more laborious) to write them as functions of the cartesian coordinates x and y:

$$H(x,y) = 1 - \frac{1}{\pi}\text{Arg}(\frac{x+iy-1}{x+iy+i}\frac{1-i}{2}) - \frac{1}{\pi}\text{Arg}(\frac{x+iy-1}{x+iy+i}\frac{1-i}{2} - 1).$$

After some calculations such an expression can be written without any mention of i or Arg, by repeatedly applying the formula $\text{Arg}(x+iy) = \arctan(y/x)$. I will not do this (somewhat tedious) calculation here, but see Section 119 of the textbook for an example.

8 *There was a typo in the question: it should have said* $\text{Im}(z) > 0$ *instead of* $\text{Re}(z) > 0$. *This is consistent with the rest of the question, which talks about boundary values in* $e^{i\theta}$ *and* $(-1, 1)$.

The idea is again to map the half-disk to a region where we know how to solve the problem. There are many possibilities but perhaps the simplest is to find a Mobius transformation which maps the half-disk to half of the upper half-plane, i.e., the positive quadrant. By visualizing how a Mobius transformation "pulls apart" the disk at its pole, we deduce that such a transformation should map:

$$1 \mapsto 0, i \mapsto 1, -1 \mapsto \infty.$$

The transformation which does this is:

$$T(z) = \frac{z-1}{z+1}\frac{i+1}{i-1} = \frac{z-1}{z+1}\frac{\sqrt{2}e^{i\pi/4}}{\sqrt{2}e^{3\pi/4}} = -i\frac{z-1}{z+1}.$$

We know that this maps $D(0,1)$ to the upper halfplane. Observe that $T(z)$ lies on the positive imaginary axis for every real $z \in (-1,1)$. This implies that the images of the top and bottom halves of $D(0,1)$ are separated by the positive imaginary axis, so they must be quadrants. To confirm that the upper half maps to the positive quadrant, we observe that i maps to 1, which lies on the boundary of the positive quadrant.

The Dirichlet problem in the quadrant $\{w : \text{Im}(w) > 0, \text{Re}(w) > 0\}$ has boundary conditions 0 on the positive imaginary axis and 1 on the positive real axis. This is easily solved by using an argument function:

$$h(w) = 1 - \frac{2}{\pi}\text{Arg}(w)$$

where $\text{Arg}(w)$ is the principal branch.

Mapping this back to a solution in the disk, we obtain

$$H(z) = h(T(z)) = 1 - \frac{2}{\pi}\text{Arg}(-i\frac{z-1}{z+1})$$

in the half-disk.