Math 185-5 Fall 2015, Homework 13

Due December 2 in class

1. The inverse function theorem says that if $f'(z_0) \neq 0$, there is a neighborhood $D(z_0, r)$ of z_0 in which f is 1-1 (i.e., for $z_1, z_2 \in D(z_0, r)$ we have $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$).

   - Show additionally that there is a neighborhood $D(w_0, \rho)$ of $w_0 = f(z_0)$ and a function $g : D(w_0, \rho) \to D(z_0, r)$ such that $g(f(z)) = z$ for all z such that $f(z) \in D(w_0, \rho)$. (hint: use the open mapping theorem)
   - Show that g is analytic at w_0 and $g'(w_0) = \frac{1}{f'(z_0)}$. (hint: go back to the definition of a derivative and change variables from w to z).

   It is possible to prove both these statements using multivariable calculus (as is outlined in Section 114 of the book), but I want you to prove them using methods of complex analysis.

2. Show that if $u(x, y)$ is harmonic in a domain then u_x has a harmonic conjugate in that domain (hint: recall that $f'(z) = u_x - iu_y$ for $f = u + iv$).

   Use this to show that if $u(x, y)$ is harmonic in a *simply connected* domain D then u has a harmonic conjugate in D. (hint: consider the antiderivative of the function in the previous part)

3. Show that if $u(x, y)$ is harmonic in a domain D then $|u(x, y)|$ does not have any local maxima. (hint: open mapping)

4. Explain why $u(x, y) = \ln(x^2 + y^2)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$ but why it cannot have a harmonic conjugate there.

5. Brown and Churchill 114.6, 114.10, 116.4, 116.5, 117.2.

6. Prove the following generalization of the inverse function theorem: if $f'(z_0) = \ldots = f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$ then there is a neighborhood of z_0 in which f is k-to-1. The following steps are suggested:

   - By a Taylor expansion write
   $$f(z) = f(z_0) + (z - z_0)^k g(z)$$
   for some $g(z_0) \neq 0$ in a neighborhood of z_0.
   - Since $g(z_0) \neq 0$ the function
   $$h(z) = \exp(\frac{1}{k}\text{Log}(g(z))$$
   is analytic and nonzero in a neighborhood of z_0, for some appropriate branch of the logarithm, and $h(z)^k = g(z)$ there. Thus,
   $$f(z) - f(z_0) = ((z - z_0)h(z))^k = \phi(z)^k$$
   where $\phi(z) = (z - z_0)h(z)$, in a neighborhood N of z_0.
   - Apply the inverse function theorem to $\phi(z)$ and consider the behavior of the kth power near zero to show that f must be k-to-1 in a neighborhood of z_0.

Note that this generalization provides a converse to the inverse function theorem: if $f : U \to V$ is analytic and 1-1 then $f'(z) \neq 0$ for all $z \in U$ (since otherwise f would be somewhere locally k-to-1 for some $k > 1$). This in turn implies that 1-1 analytic functions have analytic inverses.

(optional) There is also a proof using Rouche's theorem, if you are interested in finding it.

7. Find a function $H(x, y)$ that is harmonic in $D(0, 1)$ and extends continuously to a function with boundary values:

$$\overline{H}(e^{i\theta}) = \begin{cases} 0 & \theta \in (0, \pi) \\ 1 & \theta \in (\pi, 3\pi/2) \\ -1 & \theta \in (3\pi/2, 2\pi) \end{cases}$$

(hint: use a Möbius transformation to map it to a Dirichlet problem in the upper halfplane).

8. Find a function $H(x, y)$ harmonic in the interior of the half-disk $\{z : |z| < 1, \text{Re}(z) > 0\}$ with boundary values 1 on the semicircle $\{e^{i\theta} : \theta \in (0, \pi)\}$ and 0 on the interval $(-1, 1)$.

9. (optional) Show that every Möbius transformation which maps $D(0, 1)$ to $D(0, 1)$ is of the form

$$z \mapsto e^{i\theta} \frac{a - z}{1 - \overline{a}z},$$

for some real θ and $a \in D(0, 1)$. Recall that we proved in HW1 every such mapping does indeed map $D(0, 1)$ to $D(0, 1)$ (they are called Blaschke factors).