Math 185-5 Fall 2015, Homework 12 Selected Solutions

1a The poles of $f(z) = \frac{z^5}{1-z^3}$ are at the cube roots of 1, so f is analytic on and *outside* the contour $|z| = 2$. By Section 77 we have (taking the integral positively):

$$\oint_{|z|=2} f(z)dz = -2\pi i \text{Res}(f(z), \infty) = 2\pi i \text{Res}(-\frac{1}{z^2} f(1/z), 0).$$

The Laurent series of the latter function about zero is:

$$\frac{-1}{z^2} \frac{(1/z)^5}{1-(1/z)^3} = \frac{-1}{z^7}\left(1 + \frac{1}{z^3} + \frac{1}{z^6} + \dots\right) = \frac{-1}{z^7} + \frac{-1}{z^4} + \frac{-1}{z} + \dots,$$

so the relevant residue is -1 and the value of the integral is $-2\pi i$.

2* Since $w(z) = 1/z$ is a bijection from $\mathbb{C} \setminus \{0\}$ to itself, every nonzero point on the circle $\{z : |z-a|^2 = r^2\}$ is mapped to a unique nonzero $w = 1/z$. Thus the image of the circle possibly excluding zero:

$$C = \{z : |z-a|^2 = r^2, z \neq 0\}$$

under the inversion mapping is the set of all w satisfying:

$$|(1/w) - a|^2 = r^2 \iff (1 - wa)\overline{(1 - wa)} = |w|^2 r^2 \iff 1 - 2\text{Re}(wa) + |a|^2|w|^2 = |w|^2 r^2..$$

If $|a|^2 = r^2$ then this simplifies to $2\text{Re}(wa) = 1$, which in Cartesian coordinates $w = u + iv$ and $a = x + iy$ becomes $2(ux - vy) = 1$, which is the equation of a line. Note that in this case 0 lies on the circle and gets mapped to ∞.

If $|a|^2 \neq r^2$ then the original circle does not pass through zero, and letting $\rho = |a|^2 - r^2 \neq 0$ we rewrite the above equation in Cartesian coordinates as:

$$1 - 2(ux - vy) + (u^2 + v^2)\rho = 0 \iff u^2 - 2u\frac{x}{\rho} + v^2 + 2v\frac{y}{\rho} + \frac{1}{\rho} = 0.$$

Completing squares, we obtain

$$\left(u - \frac{x}{\rho}\right)^2 + \left(v + \frac{y}{\rho}\right)^2 = \frac{x^2 + y^2 - \rho}{\rho^2} = \frac{r^2}{\rho^2} > 0,$$

which is the equation of a circle with radius $r/|\rho|$ and center $(x/\rho, -y/\rho)$.

3* Since translations, dilations, rotations, and inversions are bijections and the composition of a finite number of bijections is a bijection, every Möbius transformation is a bijection from $\mathbb{C} \cup \{\infty\}$ to itself.

98.12* If the original circle is centered at zero it gets mapped to infinity. If not, then let $a = x + iy$ be the center of the original circle. By problem 2, the center of the image circle is $a' = x/\rho - iy/\rho$ for some $\rho \neq 0$. In order to have $a' = 1/a$ we must have $|a'| = |x - iy|/|\rho| = |a|/|\rho| = \frac{1}{|a|}$ which implies that $|\rho| = |a|^2$. But by problem 2 we have $\rho = |a|^2 - r^2$ where r is the radius of the original circle, so this is only possible if $r = 0$, which is absurd.

6* Since $\text{Im}(1) = 0$, the halfplane can be written as $H = \{z : \text{Im}(e^{i\pi/4}z) > 0\}$, which is a rotation of the upper halfplane about the origin by an angle of $\pi/4$ clockwise. To map this to the disk, we first map it to the upper half plane by the transformation

$$T_1(z) = e^{i\pi/4}z.$$

We then map the upper half plane to $D(0,1)$ by the transformation

$$T_2(z) = \frac{z - i}{z + i}$$

discussed in Section 101. Thus, the required transformation is the composition

$$T(z) = T_2 \circ T_1(z) = \frac{e^{i\pi/4}z - i}{e^{i\pi/4}z + i}.$$

7 For any halfplane $H = \{z : \text{Im}(az + b) > 0\}$ there is a translation and a rotation

$$T_1(z) = az + b$$

mapping it to the upper half plane. Composing this with the mapping $T_2(z) = \frac{z-i}{z+i}$ yields the required transformation, which is just

$$T_2 \circ T_1(z) = \frac{az + b - i}{az + b + i}.$$

To map the disk to H we simply invert this transformation:

$$w = \frac{az + b - i}{az + b + i} \iff waz + wb + wi = az + b - i \iff z = \frac{-wb - wi + b - i}{wa - a}.$$

8* There is no Möbius transformation from $\mathbb{C}$ to $D(0,1)$ because such a transformation would be an entire function whose image is contained in $D(0,1)$, i.e., a bounded entire function. But by Liouville's theorem every such function is constant.

Another proof relies on the fact that every Möbius transformation is a *bijection* from $\mathbb{C} \cup \{\infty\}$ to itself. In particular it is 1-1, so it would have to map the complement of $\mathbb{C}$ in $\mathbb{C} \cup \{\infty\}$ to the complement of $D(0,1)$ in $\mathbb{C} \cup \{\infty\}$. But the first set contains a single point, namely ∞, whereas the latter set is infinite, so this is impossible.

The second argument also implies that it is not posible for a Möbius transformation to map the complex plane minus any finite number of points to $D(0,1)$.