1a The poles of \( f(z) = \frac{z^3}{1 - z^3} \) are at the cube roots of 1, so \( f \) is analytic on and outside the contour \(|z| = 2\). By Section 77 we have (taking the integral positively):

\[
\oint_{|z|=2} f(z) dz = -2\pi i \text{Res}(f(z), \infty) = 2\pi i \text{Res}(\frac{1}{z^2} f(1/z), 0).
\]

The Laurent series of the latter function about zero is:

\[
-\frac{1}{z^2} \frac{(1/z)^5}{1 - (1/z)^3} = -\frac{1}{z^2} \left( 1 + \frac{1}{z^3} + \frac{1}{z^6} + \ldots \right) = -\frac{1}{z^2} - \frac{1}{z} + \ldots,
\]

so the relevant residue is \(-1\) and the value of the integral is \(-2\pi i\).

2* Since \( w(z) = 1/z \) is a bijection from \( \mathbb{C} \setminus \{0\} \) to itself, every nonzero point on the circle \( \{z : |z-a|^2 = r^2\} \) is mapped to a unique nonzero \( w = 1/z \). Thus the image of the circle possibly excluding zero:

\[ C = \{z : |z-a|^2 = r^2, z \neq 0\} \]

under the inversion mapping is the set of all \( w \) satisfying:

\[ |(1/w) - a|^2 = r^2 \iff (1 - wa)(1 - wa) = |w|^2 r^2 \iff 1 - 2Re(wa) + |a|^2 |w|^2 = |w|^2 r^2. \]

If \(|a|^2 = r^2\) then this simplifies to \(2\text{Re}(wa) = 1\), which in Cartesian coordinates \( w = u + iv \) and \( a = x + iy \) becomes \(2(xu - vy) = 1\), which is the equation of a line. Note that in this case 0 lies on the circle and gets mapped to \( \infty \).

If \(|a|^2 \neq r^2\) then the original circle does not pass through zero, and letting \( \rho = |a|^2 - r^2 \neq 0 \) we rewrite the above equation in Cartesian coordinates as:

\[ 1 - 2(xu - vy) + (u^2 + v^2) \rho = 0 \iff u^2 - 2ux \rho + v^2 + 2vy \rho + \frac{1}{\rho} = 0. \]

Completing squares, we obtain

\[
\left( u - \frac{x}{\rho} \right)^2 + \left( v + \frac{y}{\rho} \right)^2 = \frac{x^2 + y^2 - \rho}{\rho^2} = \frac{r^2}{\rho^2} > 0,
\]

which is the equation of a circle with radius \( r/|\rho| \) and center \((x/\rho, -y/\rho)\).

3* Since translations, dilations, rotations, and inversions are bijections and the composition of a finite number of bijections is a bijection, every Möbius transformation is a bijection from \( \mathbb{C} \cup \{\infty\} \) to itself.

98.12* If the original circle is centered at zero it gets mapped to infinity. If not, then let \( a = x + iy \) be the center of the original circle. By problem 2, the center of the image circle is \( a' = x/\rho - iy/\rho \) for some \( \rho \neq 0 \). In order to have \( a' = 1/a \) we must have \(|a'| = |x - iy|/|\rho| = |a|/|\rho| = \frac{1}{|a|} \) which implies that \(|\rho| = |a|^2\). But by problem 2 we have \( \rho = |a|^2 - r^2 \) where \( r \) is the radius of the original circle, so this is only possible if \( r = 0 \), which is absurd.
Since \( \text{Im}(1) = 0 \), the halfplane can be written as \( H = \{ z : \text{Im}(e^{i\pi/4}z) > 0 \} \), which is a rotation of the upper halfplane about the origin by an angle of \( \pi/4 \) clockwise. To map this to the disk, we first map it to the upper half plane by the transformation

\[
T_1(z) = e^{i\pi/4}z.
\]

We then map the upper half plane to \( D(0,1) \) by the transformation

\[
T_2(z) = \frac{z - i}{z + i}
\]

discussed in Section 101. Thus, the required transformation is the composition

\[
T(z) = T_2 \circ T_1(z) = \frac{e^{i\pi/4}z - i}{e^{i\pi/4}z + i}.
\]

For any halfplane \( H = \{ z : \text{Im}(az + b) > 0 \} \) there is a translation and a rotation

\[
T_1(z) = az + b
\]

mapping it to the upper half plane. Composing this with the mapping \( T_2(z) = \frac{z - i}{z + i} \) yields the required transformation, which is just

\[
T_2 \circ T_1(z) = \frac{az + b - i}{az + b + i}.
\]

To map the disk to \( H \) we simply invert this transformation:

\[
w = \frac{az + b - i}{az + b + i} \iff waz + wb + wi = az + b - i \iff z = \frac{-wb - wi + b - i}{wa - a}.
\]

There is no Möbius transformation from \( \mathbb{C} \) to \( D(0,1) \) because such a transformation would be an entire function whose image is contained in \( D(0,1) \), i.e., a bounded entire function. But by Liouville’s theorem every such function is constant.

Another proof relies on the fact that every Möbius transformation is a \textit{bijection} from \( \mathbb{C} \cup \{ \infty \} \) to itself. In particular it is 1-1, so it would have to map the complement of \( \mathbb{C} \) in \( \mathbb{C} \cup \{ \infty \} \) to the complement of \( D(0,1) \) in \( \mathbb{C} \cup \{ \infty \} \). But the first set contains a single point, namely \( \infty \), whereas the latter set is infinite, so this is impossible.

The second argument also implies that it is not possible for a Möbius transformation to map the complex plane minus any finite number of points to \( D(0,1) \).