Math 185-5 Fall 2015, Homework 11 Solutions

1. Observe that $f(z) = z^{-2}\pi/\sin(\pi z)$ has simple poles at the nonzero integers since $\sin(\pi z)$ has simple zeros there, and a triple pole at $z = 0$. The residues at the nonzero poles are:

$$\mathrm{Res}(f, n) = \lim_{z\to n} \frac{\pi(z-n)}{z^2\sin(\pi(z-n)+n\pi)} = \frac{(-1)^n}{n^2},$$

which are exactly the summands of the series we are interested in.

As in class, the goal is to find contours which enclose poles corresponding to the partial sums $S_N = \sum_{n=1}^{N}(-1)^2/n^2$ as well as the pole at zero, and on which the function is small (ideally vanishing) so we can easily compute the integral. The same contours B_N, given by the axis-aligned box with sides through the points $\pm N/2, \pm iN/2$ where N is an odd integer, will work.

Let us show that

$$|1/\sin(\pi z)| = \frac{2}{|e^{i\pi z} - e^{-i\pi z}|} < 4$$

whenever $z \in B_N$ for $N > 2$. Let $z = x + iy$. There are two cases:

(1) $y = \pm N/2$. Here $|e^{-i\pi(x+iy)}| = e^{\pi y} > e^{\pi}$ and $e^{i\pi(x+iy)} = e^{-\pi y} < e^{-\pi}$ since $N/2 > 1$. Thus, the quantity we are interested in is bounded by

$$\frac{2}{e^{\pi} - e^{-\pi}} < 4.$$

The case $y = -N/2$ is completely analogous with the roles of y and $-y$ switched.

(2) $x = N/2$. Notice that $e^{iz} - e^{-iz} = e^{-\pi y}e^{i\pi x} - e^{\pi y}e^{-i\pi x}$. Since x is a half-integer both $e^{i\pi x}$ and $e^{-i\pi x} = \overline{e^{i\pi x}}$ are pure imaginary with the *opposite sign*. Thus, we have

$$|e^{i\pi z} - e^{-i\pi z}| = |e^{-\pi y} + e^{\pi y}| \geq 1,$$

so we are done.

As a result we have $|f(z)| < 4\pi/|z|^2 \leq 4\pi/N^2$ on B_N whenever $N > 2$ and N is odd. Thus, we have for every odd N by the ML estimate:

$$2\sum_{n=1}^{(N-1)/2} \mathrm{Res}(f, n) + \mathrm{Res}(f, 0) = 2\pi i \oint_{B_N} f(z)dz \leq 2\pi i \cdot 4N \cdot 4\pi/N^2 = O(1/N).$$

(The factor 2 appears because every residue is counted twice, once at n and once at $-n$).

Taking a limit as $N \to \infty$ (by a sequence of odd integers), we have

$$2\sum_{n=1}^{\infty}(-1)^n/n^2 + \mathrm{Res}(f, 0) = 0,$$

so the value of the infinite sum is $-\mathrm{Res}(f, 0)/2$.

We calculate this via a Laurent series expansion at zero:

$$\frac{\pi}{z^2}\frac{1}{\pi z(1-(\pi z)^2/3!+O(z^4))}=\frac{1}{z^3}\left(1+(\pi z)^2/3!+O(z^4)\right)=\frac{1}{z^3}+\frac{\pi^2}{6z}+\dots$$

So we have $S=-\pi^2/12$.

2. (1) $f(z)$ has simple poles at the ath and bth roots of unity (which are distinct since a and b are relatively prime). To calculate the residue at $e^{2\pi ik/a}$, where $k=0,\dots,a-1$, it is easiest to use the following trick which appears in Section 78 of the book: If $f(z)=p(z)/q(z)$ has a simple pole at z_0 with $q(z_0)=0$ and $p(z_0)\neq 0$ then

$$\mathrm{Res}(f,z_0)=\lim_{z\to z_0}\frac{p(z)(z-z_0)}{q(z)}=\lim_{z\to z_0}\frac{p(z)(z-z_0)}{q(z)-q(z_0)}=\frac{p(z_0)}{q'(z_0}.$$

Applying this to our function at $e^{2\pi ik/a}$ with $p(z)=\frac{1}{z^{t+1}(1-z^b)}$ and $q(z)=(1-z^a)$ we have

$$\mathrm{Res}(f,e^{2\pi ik/a})=\frac{1}{(-a(e^{2\pi ik/a})^{a-1})(1-e^{2\pi ikb/a})e^{2\pi ik(t+1)/a}}=\frac{1}{(-ae^{-2\pi ik/a})(1-e^{2\pi ikb/a})e^{2\pi ik(t+1)/a}}.$$

The formula for $\mathrm{Res}(f,b)$ is identical with the roles of a and b reversed.

(2) We will calculate the residue at zero by calculating the Laurent expansion at zero, which by expanding $1/(1-z^a)$ and $1/(1-z^b)$ as geometric series for $|z|<1$ is:

$$\frac{1}{z^{t+1}}\left(\sum_{n=0}^{\infty}z^{an}\right)\left(\sum_{m=0}^{\infty}z^{bn}\right)=\frac{1}{z^{t+1}}\sum_{\ell=0}^{\infty}\sum_{an+bm=\ell,m,n\geq 0}z^{\ell}=\frac{1}{z^{t+1}}\sum_{\ell=0}^{\infty}N(\ell)z^{\ell}.$$

Thus, the residue at zero is the coefficient of z^t in the power series, which is $N(t)$.

(3) Since the denominator is a polynomial of degree $a+b+t+1\geq 2$, we know that $\max_{|z|=R}f(z)=O(1/R^2)$ for a large circle C_R centered at the origin. Applying an ML estimate to $\oint_{C_R}f(z)dz$ and taking a limit as $R\to\infty$ shows that

$$0=\frac{1}{2\pi i}\oint_{C_R}f(z)dz=\mathrm{Res}(f,0)+\sum_{k=0}^{a-1}\mathrm{Res}(f,e^{2\pi ik/a})+\sum_{k=0}^{b-1}\mathrm{Res}(f,e^{2\pi ik/b}).$$

Rearranging gives a formula for $N(t)$. This formula is simplified in the next part of the problem in an easy to apply form, but notice that it is already nontrivial in the sense that it only contains $a+b$ terms (no matter how large t is).

3. We would like to find α and β so that

$$R(z)=\frac{\alpha}{z-a}+\frac{\beta}{z-b}.$$

(Since a and b are distinct we know from the partial fraction decomposition that such α,β exist). Let C_a be a positively oriented circle containing a and not b and let b be such a circle around b. Observe that since $R(z)$ has only a simple pole at a inside C_a:

$$2\pi i\mathrm{Res}(R,a)=\oint_{C_a}R(z)dz=\oint_{C_a}\frac{\alpha}{z-a}dz+\oint_{C_a}\frac{\beta}{z-b}dz=\alpha\oint_{C_a}\frac{1}{z-a}dz+0,$$

since the second function is analytic on and inside C_a. The latter expression is just $2\pi i\alpha$, so we must have $\alpha=\mathrm{Res}(R,z)$. A similar argument works for β.