Math 185 HW10 Solutions

92.2

$$I = \int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$$

Let $C: z = e^{i\theta}$ $\theta \in [-\pi,\pi]$

$dz = ie^{i\theta}d\theta = iz\,d\theta$

Then $\sin\theta = \frac{z-z^{-1}}{2i}$.

So:

$$I = \oint_C \frac{1}{4\left(\frac{z-z^{-1}}{2i}\right)^2} \frac{dz}{iz} = \frac{1}{i}\oint_C \frac{dz}{\left(1-\frac{1}{4}\left(\frac{z^2-1}{z}\right)^2\right)z}$$

$$= \frac{1}{i}\oint_C \frac{4z\,dz}{4z^2-(z^2-1)^2} = \frac{-4}{i}\oint \frac{z\,dz}{z^4-6z^2-1}$$

The denominator has zeros $z^2 = \frac{6 \pm \sqrt{36-4}}{2}$

$= 3+\sqrt{8},\ 3-\sqrt{8} =: \alpha, \beta.$

So the integrand $f(z) = \frac{z}{(z^2-\alpha)(z^2-\beta)}$ has four simple poles at

$\pm\sqrt{\alpha}, \pm\sqrt{\beta}$, with only two $(\pm\sqrt{\beta})$ inside C. The corresponding

Sum of the residues is

$$\mathrm{Res}(f,\sqrt{\beta}) + \mathrm{Res}(f,-\sqrt{\beta}) = \left.\frac{\sqrt{\beta}\,(z-\sqrt{\beta})}{(z^2-\alpha)(z^2-\beta)}\right|_{z=\sqrt{\beta}} + \left.\frac{-\sqrt{\beta}\,(z+\sqrt{\beta})}{(z^2-\alpha)(z^2-\beta)}\right|_{-\sqrt{\beta}}$$

$$= \frac{\cancel{\sqrt{\beta}}}{(\beta-\alpha)(2\cancel{\sqrt{\beta}})} + \frac{-\cancel{\sqrt{\beta}}}{(\beta-\alpha)(-2\cancel{\sqrt{\beta}})} = \frac{1}{\beta-\alpha} = \frac{-1}{2\sqrt{8}}.$$

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By the residue theorem,

$$I = -\frac{4}{\cancel{i}} \cdot 2\pi \cancel{i} \frac{(-1)}{2\sqrt{8}} = \frac{4\pi}{2\sqrt{2}} = \underline{\underline{\sqrt{2}\pi}}.$$

88.2

$$I = \int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\,p.v. \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx$$ since the integrand is even.

$$= \frac{1}{2}\,p.v.\, Re \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx.$$ Let $f(z) = \dfrac{e^{iaz}}{1+z^2}$.

Let C_R be a semicircular contour at radius R centered at the origin, positively oriented.

Then:

$$\oint f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

By the ML estimate and the fact that $|e^{iaz}| = |e^{-a\,Im(z)}| \leq 1$ in the upper half plane we have

$$\int_{C_R} f(z)\,dz \leqslant 2\pi R \cdot \frac{1}{R^2-1} \longrightarrow 0 \text{ as } R \to \infty$$

Also, the integrand has poles at $\pm i$; by the Residue thm

$$\int_{\triangle} f(z)\,dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \frac{e^{iaz}}{\cancel{(z-i)}(z+i)} \cancel{(z-i)} \Big|_{z=i}$$

$$= 2\pi i \frac{e^{-a}}{2i} = \pi e^{-a}.$$

Taking a limit as $R \to \infty$:

$$\pi e^{-a} = \int_{-\infty}^{\infty} \frac{e^{iaz}}{1+z^2}\,dz$$

So $2I = \operatorname{Re}(\pi e^{-a}) \Rightarrow I = \frac{\pi e^{-a}}{2}$, as desired.

88.12

(a) $f(z) = e^{iz^2}$ is entire. Let:

C_1: $z(x) = x$, $x \in [0, R]$

C_R: $z(\theta) = Re^{i\theta}$ $\theta \in [0, \pi/4]$

C_2: $z(r) = (R-r)e^{i\pi/4}$, $r \in [0, R]$:

By Cauchy Goursat:

$$0 = \operatorname{Re} \int_{C_1 + C_R + C_2} f(z)\,dz = \int_{C_1} \operatorname{Re} f(z)\,dz + \operatorname{Re} \int_{C_R} f(z)\,dz + \operatorname{Re} \int_{C_2} f(z)\,dz$$

$$= \int_0^R \cos(x^2)\,dx + \operatorname{Re} \int_{C_R} e^{iz^2}\,dz + \int_0^R \operatorname{Re}\left[e^{i(R-r)^2 \overbrace{e^{i\pi/2}}^{i}} - e^{i\pi/4}\right] dr$$

$$= \int_0^R \cos(x^2)\,dx + \mathrm{Re}\int_{C_R} e^{iz^2}\,dz + \int_0^R e^{-(R-r)^2}\cos(-\pi/4)\,dr \,.$$

$$= \int_0^R \cos(x^2)\,dx + \mathrm{Re}\int_{C_R} e^{iz^2}\,dz - \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr$$

change of vars, $R-r \to r$

Rearranging gives the desired equality.

The calculation is completely analogous for $\sin(r^2)$, but with Im instead of Re.

(b) By the triangle inequality for integrals of functions of a real variable:

$$\left| \int_{C_R} e^{iz^2}\,dz \right| = \left| \int_0^{\pi/4} e^{iR^2 e^{i2\theta}}\, iRe^{i\theta}\,d\theta \right|$$

$$\leq \int_0^{\pi/4} \left| e^{iR^2e^{i2\theta}} \right| \left| iRe^{i\theta} \right| d\theta = \int_0^{\pi/4} Re^{-R^2\sin 2\theta}\,d\theta$$

$$= \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi \,. \qquad \begin{pmatrix} \phi = 2\theta \\ d\phi = 2\,d\theta \end{pmatrix}$$

$$\leq \frac{R}{2}\int_0^{\pi} e^{-R^2\sin\phi}\,d\phi \quad \text{since } e^{-R^2\sin\phi} \geq 0$$

$$\leq \frac{R}{2}\cdot\frac{\pi}{R^2} \text{ by Jordan's Lemma } \leq \frac{\pi}{2R} \longrightarrow 0 \text{ as } R \to \infty.$$

(c) Taking a limit of (a) as $R \to \infty$ we have

$$\int_0^\infty \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^\infty e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

and similarly for $\int_0^\infty \sin(x^2)\,dx$.

(By the way, these integrals are related to how people design the shapes of rollercoasters; see

http://thatsmaths.com/2014/04/10/rollercoaster-loops/

for details .)

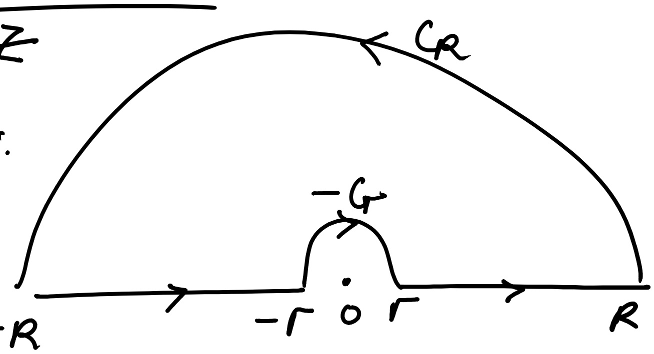
91.1

Assume $a \neq b$ (otherwise the integrand is zero and the formula holds trivially). The function $f(z)$ has a <u>simple pole</u> at $z=0$ since

$$\frac{e^{iaz}-e^{ibz}}{z^2} = \frac{z(a-b) + \frac{z^2}{2!}(a^2-b^2) + \cdots}{z^2}$$

$$= \frac{(a-b) + \frac{z}{2!}(a^2-b^2) + \cdots}{z}$$ and the numerator is nonzero.

Consider the contour C:



With semicircles: C_R: $Re^{i\theta}$ $\theta \in [0,\pi]$

C_r: $re^{i\theta}$ $\theta \in [0,\pi]$.

Since $f(z)$ is analytic on/inside C, we have

$$0 = \int_{-R}^{-r} f(z)\,dz + \int_{-C_r} f(z)\,dz + \int_{r}^{R} f(z)\,dz + \int_{C_R} f(z)\,dz$$

Since $\operatorname{Re} f(-z) = \cos(-z) = \cos(z) = \operatorname{Re} f(z)$, we have

$$\operatorname{Re}\int_{-R}^{-r} f(z)\,dz = -\operatorname{Re}\int_{R}^{r} f(-z)\,dz = \operatorname{Re}\int_{r}^{R} f(z)\,dz .$$

Thus, rearranging and taking real parts gives:

$$2\,\mathrm{Re}\int_r^R f(z)\,dz = -\mathrm{Re}\int_{C_R} f(z)\,dz - \mathrm{Re}\int_{-C_r} f(z)\,dz .$$

We will take limits as $R \to \infty$ and $r \to 0$.

By the ML estimate and the boundedness of e^{iaz}, e^{ibz} for $\mathrm{Im}(z) \geq 0$ since $a, b \geq 0$, we have

$$\lim_{R\to\infty} \left| \int_{C_R} f(z)\,dz \right| \leq \lim_{R\to\infty} \left| \int_{C_R} \frac{e^{iaz}}{z^2}\,dz \right| + \left| \int_{C_R} \frac{e^{ibz}}{z^2}\,dz \right|$$

$$\leq \lim_{R\to\infty} \frac{\pi R}{R^2} + \frac{\pi R}{R^2} = 0.$$

By the Theorem in Sec 89 (or by direct computation), we have

$$\lim_{r\to 0} \int_{-C_r} f(z)\,dz = -i\pi\,\mathrm{Res}(f,0) = -i\pi(a-b)$$

from the power series expansion.

Thus, we have

$$2 \lim_{r\to 0} \lim_{R\to\infty} \mathrm{Re}\int_r^R f(z)\,dz = 0 + \pi(b-a)$$

So $$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \mathrm{Re}\int_0^\infty f(z)\,dz = \frac{\pi(b-a)}{2}$$

Taking $a = 0$ and $b = 2$ gives

$$\int_0^\infty \frac{\cos(0) - \cos 2x}{x^2}\,dx = \int_0^\infty \frac{2\sin^2 x}{x^2}\,dx = \pi,$$

as desired.

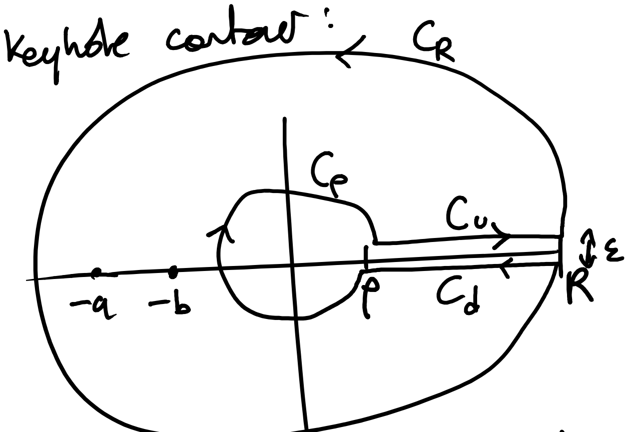
<u>91.4</u>

Let $I = \displaystyle\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx$, and let

$f(z) = \dfrac{\exp(\frac{1}{3}\log z)}{(z+a)(z+b)}$ where we take $0 < \arg(z) < 2\pi$.

$f(z)$ has a branch cut on the nonnegative real axis and isolated simple poles at $z = -a, -b$.

Consider the keyhole contour:



Optional justification

The contours C_u, C_d can be taken arbitrarily close to the branch cut, say at distance ε. Since the integrand is continuous above and below the branch cut and C_u, C_d are compact we have

$$\lim_{\varepsilon \to 0} \int_{C_u} f(z)\,dz = \int_\rho^R \frac{\exp(\frac{1}{3}\ln x + 0i)}{(x+a)(x+b)}\,dx = I.$$

$$\lim_{\varepsilon\to 0}\int_{C_d} f(z)\,dz = -\int_{\rho}^{R}\frac{\exp\left(\frac{1}{3}(\ln x + 2\pi i)\right)}{(x+a)(x+b)}\,dx$$

$$= -e^{\frac{2\pi i}{3}}\int_{\rho}^{R}\frac{\exp\left(\frac{1}{3}\ln x\right)}{(x+a)(x+b)}\,dx = -e^{\frac{2\pi i}{3}} I.$$

On the other hand, the Residue theorem implies that

$$\left(\int_{C_d} + \int_{C_v} + \int_{C_R} + \int_{C_\rho}\right) f(z)\,dz = 2\pi i\left(\operatorname{Res}(f,-a) + \operatorname{Res}(f,-b)\right).$$

These residues are given by: $\operatorname{Res}(f,-a) = \dfrac{\exp\left(\frac{1}{3}\operatorname{Log}(-a)\right)}{(-a+b)}$

$$= \frac{\exp\left(\frac{1}{3}(\ln a + i\pi)\right)}{b-a}$$

and similarly: $\operatorname{Res}(f,-b) = \dfrac{\exp\left(\frac{1}{3}(\ln b + i\pi)\right)}{-b+a}$

We also have

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| \le \lim_{R\to\infty} \pi R\, O\left(\frac{R^{1/3}}{R^2}\right) = 0$$

$$\lim_{\rho\to 0}\left|\int_{C_\rho} f(z)\,dz\right| \le \lim_{\rho\to 0} \pi\rho\cdot\frac{\rho^{1/3}}{(a+o(1))(b+o(1))} = 0$$

Taking limits as $\varepsilon \to 0$, $R \to \infty$, $\rho \to 0$ (in that order) we have:

$$\left(1 - e^{\frac{2\pi i}{3}}\right) I = 2\pi i \left(e^{i\pi/3} \frac{\sqrt[3]{a}}{b-a} + e^{i\pi/3} \frac{\sqrt[3]{b}}{a-b} \right)$$

So
$$I = \underbrace{\frac{2\pi i \, e^{i\pi/3}}{1 - e^{2\pi i/3}}} \left(\frac{\sqrt[3]{a}}{b-a} - \frac{\sqrt[3]{b}}{b-a} \right)$$

The constant is:
$$\frac{2\pi i}{e^{-i\pi/3} - e^{i\pi/3}} = \frac{\pi}{-\sin(\pi/3)}$$
$$= \frac{2\pi}{\sqrt{3}}$$

which is the desired formula