# LECTURE NOTES 1: A PROOF OF THE VALUE OF $\zeta(2)$

## 1. The Riemann zeta function

The Riemann zeta function $\zeta : (1, \infty) \to (0, \infty)$ is defined by the formula

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}\,.$$

The sum on the right-hand side is by definition the limit $\lim_n \sum_{k=1}^{n} k^{-s}$ of the partial sums; the limit exists (though it may be $\infty$) because the partial sums form a non-decreasing sequence of (positive) real numbers. In fact, for any $s > 1$, the expression defining $\zeta(s)$ is finite: this is true by the integral test.

All of these things are true by the ideas and techniques of real analysis in Math 104; in fact, the Riemann zeta function has a natural definition as a complex-valued function of the complex variable $s$ (except for certain special values of $s \in \mathbb{C}$ called the *poles* of $\zeta$). Later on in the course, we will set up much of the machinery to define $\zeta$ in this more general way, whether or not we have time to construct it.

Partly in order to illustrate the usefulness of complex numbers, we will now prove a theorem which identifies the value of $\zeta(s)$ when $s = 2$:

**Theorem 1.1.** *The value of $\zeta(2)$ is $\pi^2/6$.*

There are certain preliminaries needed to prove the theorem.

## 2. Preliminaries

### 2.1. The Binomial theorem.

**Theorem 2.1.** *Let $n \in \mathbb{N}$ and let $x, y \in \mathbb{C}$. Then*

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\,,$$

*where the* binomial coefficient $\binom{n}{k}$ *is given by*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1}$$

*for $n, k \in \mathbb{N}$ with $n \geq k$; we adopt the convention that $n! = 0$.*

**Proof.** The result is proved by an induction on $n \in \mathbb{N}$; both sides are equal to one if $n = 0$, so that's the initial step for the induction verified. The generic step of the induction, for the value $n + 1$, invokes the inductive hypothesis for value $n$, as well as the identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}\,.$$

This can be verified from the definition (1), (but there is also a nice combinatorial interpretation, and it is also worthwhile to bear in mind how Pascal's triangle is formed). □

**Remark.** Note also that $\binom{n}{k} = \binom{n}{n-k}$.

2.2. **Polynomials and their roots.** A polynomial $P$ of degree $n$ is an expression $ax^n + bx^{n-1} + \cdots + C$. A root of the polynomial $P$ is a value $x$ such that $P(x) = 0$. (These roots may be real numbers: for example, the degree two or quadractic polynomial $X^2 - 1$ are has two real roots, $-1$ and 1; however, the roots can also be complex numbers. The polynomial $x^2+1$ has roots $i$ and $-i$).

There are two assertions we'll need about polynomials.

**A polynomial of degree $n \in \mathbb{N}$ has at most $n$ roots.** This is a consequence of the *Euclidean algorithm*: you may wish to recall or study this, or it is permissible to take the assertion on faith. (Later on in the course, we will be able to prove the *fundamental theorem of algebra*, which asserts that a degree-$n$ polynomial in fact has exactly $n$ roots, when these are counted according to multiplicity). □

**The formula for the sum of a polynomial's roots.** Suppose that a degree-$n$ polynomial $P(x) = ax^n + bx^{n-1} + \cdots + C$ is factored as $P(x) = a(x - r_1) \cdot \cdots \cdot (x - r_n)$, where $r_1, \ldots, r_n$ are the roots of $P$. Then

$$\sum_{k=1}^{n} r_k = \frac{-b}{a} \,.$$

Let's prove this first when $a = 1$. Here we just compare the coefficient of $x^{n-1}$ in the two expressions for $P$. The case where $a$ is any non-zero complex number reduces to this one by defining $P$ through by $a$. □

2.3. **Some basic properties of trigonometry functions.** The functions sin and cosine, $\sin, \cos : \mathbb{R} \to [-1, 1]$ are defined on the real line and have the property that sin is strictly increasing on $[0, \pi/2]$ and cos is strictly decreasing this interval. We also define $\tan = \sin / \cos$ and $\cot = \cos / \sin$, as well as $\csc = 1/\sin$.

From the above, we see that cot **is a strictly decreasing function on** $(0, \pi/2)$**.**

Of course we have $\cos^2(\theta) + \sin^2(\theta) = 1$ for all $\theta \in \mathbb{R}$. Dividing through by $\sin^2(\theta)$, we obtain the identity

$$1 + \cot^2(\theta) = \csc^2(\theta) \text{ for all } \theta \in \mathbb{R} \,, \tag{2}$$

(though note that sometimes both sides are $\infty$.)

Finally, we will make use of the fact that

$$\sin\theta \leq \theta \leq \tan\theta$$

whenever $\theta \in (0, \pi/2)$.

## 3. Proof of Theorem 1.1

We begin by deriving an interesting identity.

**Proposition 3.1.** *For any $n \in \mathbb{N}$ with $n \geq 1$,*

$$\sum_{k=1}^{n} \cot^2 \tfrac{k\pi}{2n+1} = \frac{n(2n-1)}{3} \,.$$

**Proof.** The proof is an application of complex numbers. Consider the expression $\big(\cos\theta + i\sin\theta\big)^{2n+1}$. Using the polar coordinate representation to multiply a pair of complex numbers, and induction on $n$, we find that

$$\big(\cos\theta + i\sin\theta\big)^{2n+1} = \cos(2n+1)\theta + i\sin(2n+1)\theta \,.$$

Expand the left-hand side using the binomial theorem. The resulting terms are either real or imaginary; group them accordingly. Then equate the imaginary term with its counterpart above. We find that

$$\sin(2n+1)\theta = \sum_{k=0}^{n}(-1)^k \cos^{2n-2k}(\theta)\sin^{1+2k}(\theta)\binom{2n+1}{1+2k}.$$

By rewriting the right-hand side using $1+2k = 2n+1-2(n-k)$, we obtain the identity

$$\sin(2n+1)(\theta) = \sin^{2n+1}(\theta)\sum_{k=0}^{n}(-1)^k \cot^{2(n-k)}(\theta)\binom{2n+1}{2(n-k)},$$

which may be rewritten

$$\sin(2n+1)(\theta) = \sin^{2n+1}(\theta)\sum_{k=0}^{n}(-1)^{n-k}\cot^{2k}(\theta)\binom{2n+1}{2k}.$$

Now on the left-hand side of this last identity, the function has zeros (or roots) at each of $\left\{\frac{k\pi}{2n+1} : k = 0,\ldots,n\right\}$. If we now $P$ to be the degree-$n$ polynomial given by

$$P(x) = \sum_{k=0}^{n}(-1)^k x^k \binom{2n+1}{2k},$$

we learn that the polynomial $P(x^2)$ has roots at each of the values in the set $\left\{\cot\frac{k\pi}{2n+1} : k = 1,\ldots,n\right\}$: indeed, the factor $\sin^{2n+1}(\theta)$ is non-zero for all $\theta \in (0,\pi/2)$, (though it is zero when $\theta = 0$ and for this reason, we cannot include $k = 0$ here).

That means each element of the set $\left\{\cot^2\frac{k\pi}{2n+1} : k = 1,\ldots,n\right\}$ is in fact a root of $P$ itself. Now we claim that this exhausts the set of roots of $P$. To see this, first note that each of these $n$ real numbers is distinct: indeed, because the function cot is strictly decreasing on $(0,\pi/2)$, so is $\cot^2$, so the $n$ numbers must all be different. Second, we noted in the preliminaries that a degree-$n$ polynomial such as $P$ cannot have more than $n$ roots. So our list of $n$ numbers must comprise all of $P$'s roots!

This deduction allows us to use the formula for the sum of the roots of $P$. Doing so gives that

$$\sum_{k=1}^{n}\cot^2\frac{k\pi}{2n+1} = \frac{\binom{2n+1}{2(n-1)}}{\binom{2n+1}{2n}},$$

whose right-hand side may more simply be written as

$$\frac{\binom{2n+1}{3}}{\binom{2n+1}{1}}.$$

This ratio equals $\frac{n}{3}(2n-1)$. This completes the proof of Proposition 3.1. □

We also need a straightforward consequence of Proposition 3.1.

**Corollary 3.2.** *For $n \in \mathbb{N}$ with $n \geq 1$,*

$$\sum_{k=1}^{n}\csc^2\frac{k\pi}{2n+1} = \frac{2n(n+1)}{3}.$$

**Proof.** This follows directly from Proposition 3.1 and the identity 2. □

Armed with the identites in the proposition and its corollary, we are ready to give the proof of Theorem 1.1. The inequalities (valid for $\theta \in (0,\pi/2)$),

$$\sin\theta \leq \theta \leq \tan\theta$$

that we recorded earlier translate to

$$\cot\theta \leq \theta^{-1} \leq \csc\theta\,.$$

Consider the $n$ sets of inequalities given by taking $\theta = \frac{k\pi}{2n+1}$ for $k \in \{1, \ldots, n\}$; sum them up. Using the two identities we derived in the proposition and the corollary, we learn that

$$\frac{n(2n-1)}{3} \leq \sum_{k=1}^{n} \tfrac{(2n+1)^2}{k^2\pi^2} \leq \frac{2n(n+1)}{3}\,.$$

Rearranging, we see that

$$\frac{n(2n-1)}{3(2n+1))^2}\pi^2 \leq \sum_{k=1}^{n} \tfrac{1}{k^2} \leq \frac{2n(n+1)}{3(2n+1)^2}\pi^2\,.$$

The lower and upper bounds each take the form $\frac{\pi^2}{6}$ plus a certain error; in the lower bound on the left, this error cannot be smaller than a large constant times $n^{-1}$, while in the upper bound on the right, it cannot be larger than a large constant times $n^{-1}$. In this way, we see that the $n$-th partial sum for $\zeta(2)$ is sandwiched between two sequences of real numbers, each of which converges to $\pi^2/6$. So the partial sums converge to $\pi^2/6$, and we have proved Theorem 1.1. $\square$