1. (12 points) True or False (no need for justification):

(a) If \( f(z) \) is analytic and bounded in the right half plane \( H = \{ z : \text{Re}(z) > 0 \} \) then \( f \) must be constant in \( H \).
False. Consider \( e^{-z} \), whose magnitude is less than one in the right half-plane.

(b) If \( u(x,y) \) is harmonic in \( \mathbb{R}^2 \) and there is an \( M > 0 \) such that \( |u(x,y)| \leq M \) for all \( x, y \) then \( u \) must be constant.
True.

(c) \( f(z) = e^z \) is one to one in the unit disk \( D(0,1) \).
True. The disk is contained in the strip \( \{ \text{Im}(z) \in (-\pi,\pi]\} \), on which \( e^z \) is 1-1 with inverse \( \text{Log}(z) \).

(d) If \( f(z) \) has a pole of order \( m \) at \( z_0 = 0 \) then \( f(z^2) \) has a pole of order 2\( m \) at \( z_0 = 0 \).
True. \( f(z) = \frac{g(z)}{z^m} \) with analytic \( g(0) \neq 0 \), so \( f(z^2) = \frac{g(z^2)}{z^{2m}} \) and \( g(0^2) = g(0) \neq 0 \). You can also look at the Laurent series.

(e) If \( f(z) \) has a removable singularity at \( z_0 \) then \( \text{Res}(f,z_0) = 0 \).
True, by definition.

(f) If \( f \) is analytic and nonconstant in \( D(0,1) \) and \( |f(z)| \leq 1 \) for all \( z \in D(0,1) \) then \( |f(z)| < 1 \) for all \( z \in D(0,1) \).
True. Maximum modulus principle.

2. (10 points) Suppose \( u(x,y) \) and \( v(x,y) \) are harmonic in a domain \( D \) and \( v \) is the harmonic conjugate of \( u \). (i) Prove that \( u^2 - v^2 \) is harmonic in \( D \). (ii) Prove that the partial derivative \( u_x \) is harmonic in \( D \).
Solution: (i) \( f = u + iv \) is analytic in \( D \), so \( f^2 = (u + iv)^2 = u^2 - v^2 + 2iuv \) is also analytic, and its real part must be Harmonic.
(ii) \( f'(z) \) is also analytic in \( D \) and \( f'(x + iy) = u_x - iu_y \), so \( u_x \) is harmonic.

3. (10 points) (i) Find a Möbius transformation which maps the open half-disk
\[
S = \{ z : |z| < 1, \text{Im}(z) > 0 \}
\]
to a quadrant. (ii) Find a 1-1 conformal mapping of the quadrant to the upper halfplane. (iii) Find a Möbius transformation mapping the upper halfplane to the unit disk \( D = \{ z : |z| < 1 \} \). (iv) Compose these to give a 1-1 conformal map of the half-disk to the unit disk.
Justify each step (i.e., explain why the transformations you produce have the desired properties).

What goes wrong if you try to directly use the transformation \( w = z^2 \) to map \( S \) conformally and 1-1 onto \( D \)?

Solution: (i) \( w = T_1(z) = -\frac{z-1}{z+1} \) maps \( S \) to \( Q = \{ z : \text{Im}(z) > 0, \text{Re}(z) > 0 \} \). See HW13.8 for details. (ii) The mapping \( T_2(w) = w^2 \) maps \( Q \) to the upper half plane \( H \), and is conformal in \( Q \) since \( T_2'(w) = 2w \neq 0 \) there. It is also 1−1 since each point in \( H \) has a unique square root in \( Q \) by \( \text{re}^i\theta \rightarrow \sqrt{\text{re}^i\theta}/2 \). (the other square root, which is its negative, lies in the lower half plane and therefore outside \( Q \).) (iii) \( y = T_3(w) = \frac{w-i}{w+i} \) maps \( H \) to \( D \) (see Section 101 for details, and other transformations that work).

Thus, the composition

\[
T_3 \circ T_2 \circ T_1(z) = \frac{T_1(z)^2 - i}{T_1(z)^2 + i} = \frac{(-\frac{z-1}{z+1})^2 - i}{(-\frac{z-1}{z+1})^2 + i}
\]

maps \( S \) to \( D \). It is conformal and 1 − 1 because each of \( T_1, T_2, T_3 \) has these properties, and composition preserves them.

Directly using \( w = z^2 \) does not because the image of \( S \) is the slit disk \( D \setminus [0,1) \).

4. (9 points) Classify (as removable, pole, or essential) the singularity at \( z = 0 \) of the following functions, and explain why. If it is a pole calculate the residue.

\[
\frac{\text{Log}(z + 1) \sin(z)}{z^2}, \quad e^{\sin(1/z)} \quad 1 + \frac{z}{e^z - 1},
\]

where Log is the principal branch.

Solution: (i) Removable, because \( \text{Log}(z + 1) \) and \( \sin(z) \) have zeros of order 1 at \( z = 0 \) (as evidenced by their Taylor series), which cancel the pole of order 2.

Alternatively you can just compute the limit as \( z \to 0 \) and see that it is 1, which in particular is bounded.

(ii) Essential. Considering the sequence \( z_n = \frac{2}{n\pi} \) converging to zero we find that \( e^{\sin(1/z_n)} \) alternates between \( e^0 \) and \( e^1 \), so in particular the limit as \( n \to \infty \) does not exist and the singularity cannot be removable or a pole, so it is essential.

(iii) Pole of order 1 with residue 1.

5. (10 points) Show that there exists an \( \epsilon > 0 \) such that for every polynomial \( p(z) \):

\[
\max_{|z|=1} \left| \frac{1}{z} - p(z) \right| > \epsilon.
\]

(hint: argue by contradiction.)

Solution: Assume for contradiction that for every \( \epsilon > 0 \) there is a polynomial \( p_\epsilon(z) \) with

\[
\max_{|z|=1} \left| \frac{1}{z} - p_\epsilon(z) \right| < \epsilon.
\]
We can write for every such $p_\epsilon$:

$$\oint_{|z|=1} \frac{1}{z} dz = \oint_{|z|=1} \left( \frac{1}{z} - p_\epsilon(z) \right) dz + \oint_{|z|=1} p_\epsilon(z) dz.$$

The second integral is zero because $p_\epsilon(z)$ is entire. Applying an ML estimate, we get

$$\left| \oint_{|z|=1} \frac{1}{z} dz \right| \leq \max_{|z|=1} |1/z - p_\epsilon(z)| \cdot 2\pi \leq 2\pi \epsilon.$$

Since this is true for every $\epsilon > 0$ we must have

$$\oint_{|z|=1} \frac{1}{z} dz = 0.$$

But this is false (the fundamental integral says it’s $2\pi i$).

6. (10 points) Prove or disprove: there is a function $f$ analytic in $D(0,1)$ with the property that

$$f \left( \frac{1}{n^2} \right) = \frac{1}{n^3},$$

for all integers $n > 1$.

**Solution:** This is false. The idea is that an analytic function is locally approximated by the first few terms of its Taylor series, which is a polynomial (whose degree depends on the order of the zero), and which grows as an integer power of $|z|$ in a neighborhood of zero, but the function above grows as $|z|^{3/2}$.

Here is a formal proof. Suppose for contradiction that $f$ was analytic at zero. Observe that $\lim_{n \to \infty} f(1/n^2) = 0$ so by continuity it must be the case that $f(0) = 0$. Since $f$ is analytic it has a Taylor expansion

$$f(z) = a_1 z + a_2 z^2 + O(z^3)$$

convergent in a neighborhood $D(0,\epsilon)$ of zero. If $a_1 = 0$ then we have

$$\lim_{n \to \infty} \frac{f(1/n^2)}{1/n^3} = \lim_{n \to \infty} n^3 \left( a_2/n^4 + O(1/n^6) \right) = a_2 \frac{1}{n} + O(1/n^3) = 0,$$

which is absurd since $f(1/n^2) = 1/n^3$. On the other hand if $a_1 \neq 0$ we have

$$\lim_{n \to \infty} \frac{f(1/n^2)}{1/n^3} = n^3 \left( a_1/n^2 + O(1/n^4) \right) = \lim_{n \to \infty} a_1 n + O(1/n) = \infty,$$

which is again absurd. Thus, no such function can exist.

A previous version of the solutions contained a comment about a possible alternative proof involving comparison to $g(z) = \exp(3/2 \log(z))$ via the identity theorem. In fact such an approach cannot work directly because $g(z)$ is not analytic in any neighborhood of zero, which is the limit point of the sequence $1/n^2$ on which $f$ and $g$ agree.

To make such an approach work, one has to consider the function $f^2$ and show using the identity theorem that $f^2$ must be equal to $g(z) = z^3$ on $D(0,1)$, and use this to derive a contradiction.
7. (7 points) Suppose \( f \) is analytic in the closed disk \( \overline{D}(0, 1) = \{ z : |z| \leq 1 \} \) and \( |f(z)| < 1 \) for all \( z \in D(0, 1) \). Prove that \( f \) has a unique fixed point in \( D(0, 1) \) (i.e., there is a unique \( z_0 \) such that \( f(z_0) = z_0 \)).

Solution: Observe that the function \( f(z) - z \) is analytic on the unit circle, \( \partial D(0, 1) \). Since \( |f(z)| < 1 \) we have \( 1 = |z| > |f(z)| \) on \( \partial D(0, 1) \). By Rouche’s theorem, \( f(z) - z \) has the same number of zeros in \( D(0, 1) \) as the function \( -z \), which has exactly one zero. Thus \( f \) has exactly one fixed point in \( D(0, 1) \).

8. (12 points) State and prove the Cauchy Integral Formula. (you may assume the Cauchy-Goursat theorem).

Solution: See textbook.

9. (10 points) Evaluate the integral:

\[
\int_0^\pi \frac{1}{2 + \sin(2\theta)} d\theta.
\]

Justify all steps.

Solution: Brown and Churchill Sec 92, Example 1.

10. (10 points) Evaluate the integral:

\[
\int_{-\infty}^{\infty} \frac{\cos(2x) - 1}{x^2} dx.
\]

Justify all steps.

Solution: HW10 91.1 (see online solutions).