Math 185 Fall 2015, Final Exam Solutions

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8:10am–11:00am, December 17, 2015, 9 Evans Hall

1. (12 points) True or False (no need for justification):

   (a) If $f(z)$ is analytic and bounded in the right half plane $H = \{z : \text{Re}(z) > 0\}$ then f must be constant in H.

   False. Consider e^{-z}, whose magnitude is less than one in the right halfplane.

   (b) If $u(x, y)$ is harmonic in $\mathbb{R}^2$ and there is an $M > 0$ such that $|u(x, y)| \leq M$ for all x, y then u must be constant.

   True.

   (c) $f(z) = e^z$ is one to one in the unit disk $D(0, 1)$.

   True. The disk is contained in the strip $\{\text{Im}(z) \in (-\pi, \pi]\}$, on which e^z is 1-1 with inverse $\text{Log}(z)$.

   (d) If $f(z)$ has a pole of order m at $z_0 = 0$ then $f(z^2)$ has a pole of order $2m$ at $z_0 = 0$.

   True. $f(z) = \frac{g(z)}{z^m}$ with analytic $g(0) \neq 0$, so $f(z^2) = \frac{g(z^2)}{z^{2m}}$ and $g(0^2) = g(0) \neq 0$. You can also look at the Laurent series.

   (e) If $f(z)$ has a removable singularity at z_0 then $\text{Res}(f, z_0) = 0$.

   True, by definition.

   (f) If f is analytic *and nonconstant* in $D(0, 1)$ and $|f(z)| \leq 1$ for all $z \in D(0, 1)$ then $|f(z)| < 1$ for all $z \in D(0, 1)$.

   True. Maximum modulus principle.

2. (10 points) Suppose $u(x, y)$ and $v(x, y)$ are harmonic in a domain D and v is the harmonic conjugate of u. (i) Prove that $u^2 - v^2$ is harmonic in D. (ii) Prove that the partial derivative u_x is harmonic in D.

   Solution: (i) $f = u + iv$ is analytic in D, so $f^2 = (u + iv)^2 = u^2 - v^2 + 2iuv$ is also analytic, and its real part must be Harmonic.

   (ii) $f'(z)$ is also analytic in D and $f'(x + iy) = u_x - iu_y$, so u_x is harmonic.

3. (10 points) (i) Find a Möbius transformation which maps the open half-disk

   $$S = \{z : |z| < 1, \text{Im}(z) > 0\}$$

   to a quadrant. (ii) Find a 1-1 conformal mapping of the quadrant to the upper halfplane. (iii) Find a Möbius transformation mapping the upper halfplane to the unit disk $D = \{z : |z| < 1\}$. (iv) Compose these to give a 1-1 conformal map of the half-disk to the unit disk.

Justify each step (i.e., explain why the transformations you produce have the desired properties).

What goes wrong if you try to directly use the transformation $w = z^2$ to map S conformally and 1-1 onto D?

Solution: (i) $w = T_1(z) = -i\frac{z-1}{z+1}$ maps S to $Q = \{z : \mathrm{Im}(z) > 0, \mathrm{Re}(z) > 0\}$. See HW13.8 for details. (ii) The mapping $T_2(w) = w^2$ maps Q to the upper half plane H, and is conformal in Q since $T_2'(w) = 2w \neq 0$ there. It is also $1-1$ since each point in H has a *unique* square root in Q by $re^{i\theta} \to \sqrt{r}e^{i\theta/2}$. (the other square root, which is its negative, lies in the lower half plane and therefore outside Q) (iii) $y = T_3(w) = \frac{w-i}{w+i}$ maps H to D (see Section 101 for details, and other transformations that work).

Thus, the composition

$$T_3 \circ T_2 \circ T_1(z) = \frac{T_1(z)^2 - i}{T_1(z)^2 + i} = \frac{-(\frac{z-1}{z+1})^2 - i}{-(\frac{z-1}{z+1})^2 + i}$$

maps S to D. It is conformal and $1-1$ because each of T_1, T_2, T_3 has these properties, and composition preserves them.

Directly using $w = z^2$ does not because the image of S is the slit disk $D \setminus [0, 1)$.

4. (9 points) Classify (as removable, pole, or essential) the singularity at $z = 0$ of the following functions, and explain why. If it is a pole calculate the residue.

$$\frac{\mathrm{Log}(z+1)\sin(z)}{z^2} \qquad e^{\sin(1/z)} \qquad \frac{1+z}{e^z - 1},$$

where Log is the principal branch.

Solution: (i) Removable, because $\mathrm{Log}(z+1)$ and $\sin(z)$ have zeros of order 1 at $z = 0$ (as evidenced by their Taylor series), which cancel the pole of order 2.

Alternatively you can just compute the limit as $z \to 0$ and see that it is 1, which in particular is bounded.

(ii) Essential. Considering the sequence $z_n = \frac{2}{n\pi}$ converging to zero we find that $e^{\sin(1/z_n)}$ alternates between e^0 and e^1, so in particular the limit as $n \to \infty$ does not exist and the singularity cannot be removable or a pole, so it is essential.

(iii) Pole of order 1 with residue 1.

5. (10 points) Show that there exists an $\epsilon > 0$ such that for every polynomial $p(z)$:

$$\max_{|z|=1} \left| \frac{1}{z} - p(z) \right| > \epsilon.$$

(hint: argue by contradiction.)

Solution: Assume for contradiction that for every $\epsilon > 0$ there is a polynomial $p_\epsilon(z)$ with

$$\max_{|z|=1} \left| \frac{1}{z} - p_\epsilon(z) \right| < \epsilon.$$

We can write for every such p_ϵ:

$$\oint_{|z|=1} \frac{1}{z} dz = \oint_{|z|=1} \left(\frac{1}{z} - p_\epsilon(z) \right) dz + \oint_{|z|=1} p_\epsilon(z) dz.$$

The second integral is zero because $p_\epsilon(z)$ is entire. Applying an ML estimate, we get

$$\left| \oint_{|z|=1} \frac{1}{z} dz \right| \le \max_{|z|=1} |1/z - p_\epsilon(z)| \cdot 2\pi \le 2\pi\epsilon.$$

Since this is true for every $\epsilon > 0$ we must have

$$\oint_{|z|=1} \frac{1}{z} dz = 0.$$

But this is false (the fundamental integral says it's $2\pi i$).

6. (10 points) Prove or disprove: there is a function f analytic in $D(0,1)$ with the property that

$$f\left(\frac{1}{n^2}\right) = \frac{1}{n^3},$$

for all integers $n > 1$.

Solution: This is false. The idea is that an analytic function is locally approximated by the first few terms of its Taylor series, which is a polynomial (whose degree depends on the order of the zero), and which grows as an integer power of $|z|$ in a neighborhood of zero, but the function above grows as $|z|^{3/2}$.

Here is a formal proof. Suppose for contradiction that f was analytic at zero. Observe that $\lim_{n\to\infty} f(1/n^2) = 0$ so by continuity it must be the case that $f(0) = 0$. Since f is analytic it has a Taylor expansion

$$f(z) = a_1 z + a_2 z^2 + O(z^3)$$

convergent in a neighborhood $D(0,\epsilon)$ of zero. If $a_1 = 0$ then we have

$$\lim_{n\to\infty} \frac{f(1/n^2)}{1/n^3} = \lim_{n\to\infty} n^3 \left(a_2/n^4 + O(1/n^6)\right) = a_2 \frac{1}{n} + O(1/n^3) = 0,$$

which is absurd since $f(1/n^2) = 1/n^3$. On the other hand if $a_1 \neq 0$ we have

$$\lim_{n\to\infty} \frac{f(1/n^2)}{1/n^3} = n^3 \left(a_1/n^2 + O(1/n^4)\right) = \lim_{n\to\infty} a_1 n + O(1/n) = \infty,$$

which is again absurd. Thus, no such function can exist.

A previous version of the solutions contained a comment about a possible alternative proof involving comparison to $g(z) = \exp((3/2)\mathrm{Log}(z))$ via the identity theorem. In fact such an approach cannot work directly because $g(z)$ is not analytic in any neighborhood of zero, which is the limit point of the sequence $1/n^2$ on which f and g agree.

To make such an approach work, one has to consider the function f^2 and show using the identity theorem that f^2 must be equal to $g(z) = z^3$ on $D(0,1)$, and use this to derive a contradiction.

7. (7 points) Suppose f is analytic in the closed disk $\overline{D}(0,1) = \{z : |z| \leq 1\}$ and $|f(z)| < 1$ for all $z \in \overline{D}(0,1)$. Prove that f has a unique fixed point in $D(0,1)$ (i.e., there is a unique z_0 such that $f(z_0) = z_0$).

   Solution: Observe that the function $f(z) - z$ is analytic on the unit circle, $\partial D(0,1)$. Since $|f(z)| < 1$ we have $1 = |-z| > |f(z)|$ on $\partial D(0,1)$. By Rouche's theorem, $f(z) - z$ has the same number of zeros in $D(0,1)$ as the function $-z$, which has exactly one zero. Thus f has exactly one fixed point in $D(0,1)$.

8. (12 points) State and prove the Cauchy Integral Formula. (you may assume the Cauchy-Goursat theorem).

   Solution: See textbook.

9. (10 points) Evaluate the integral:

$$\int_0^{\pi} \frac{1}{2 + \sin(2\theta)} d\theta.$$

   Justify all steps.

   Solution: Brown and Churchill Sec 92, Example 1.

10. (10 points) Evaluate the integral:

$$\int_{-\infty}^{\infty} \frac{\cos(2x) - 1}{x^2} dx.$$

   Justify all steps.

   Solution: HW10 91.1 (see online solutions).