Math 185 Fall 2015, Sample Final Exam Solutions

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1. True or false:

(a) If f is analytic in the annulus $A = \{z : 1 < |z| < 2\}$ then there exist functions g and h such that g is analytic in $|z| < 2$, h is analytic in $|z| > 1$, and $f = g + h$ in A

TRUE. We can write f as a Laurent series about zero:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{m=1}^{\infty} \frac{b_n}{z^n},$$

where both series are convergent in A. Take $g(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h(z) = \sum_{m=1}^{\infty} \frac{b_n}{z^n}$; since the region of convergence of $g(z)$ is a disk centered at zero (because it is a power series), $g(z)$ must be convergent in a disk containing A, so it must be convergent in $D(0, 2)$. By the converse to Taylor's theorem, it is analytic there.

Similarly, $h(z)$ is convergent in the complement of a closed disk centered at zero (because it is a power series in $1/z$) containing A. The smallest such region is $|z| > 1$, and by uniform convergence $h(z)$ is analytic there.

(b) There is a simple closed contour in $\mathbb{C} \setminus \{0\}$ such that

$$\oint_C \frac{1}{z^2} dz = 2\pi i$$

.

FALSE. Since $1/z^2$ has an antiderivative in $\mathbb{C} \setminus \{0\}$ every closed integral is zero. Alternatively, by the residue theorem, it has only one singularity (at $z_0 = 0$) with residue zero.

(c) If $f(z)$ and $g(z)$ have a pole at z_0 then $\text{Res}(fg, z_0) = \text{Res}(f, z_0)\text{Res}(g, z_0)$.

FALSE. For instance $f(z) = 1/z$ and $g(z) = 1 + 1/z^2$ have a pole at zero with $\text{Res}(f, 0) = 1, \text{Res}(g, 0) = 0$, but the product $f(z)g(z) = 1/z + 1/z^3$ has residue 1 at zero.

(d) Every function analytic in $D(0, 1)$ has an analytic continuation to $D(0, 2)$.

FALSE. The function $f(z) = \frac{1}{1-z}$ is analytic in $D(0,1)$ but there is no $F(z)$ analytic in $D(0,2)$ with $F(z) = f(z)$ in $D(0,1)$. The reason is that for a sequence, such as $z_n = 1 - 1/n$, approaching the pole $z_0 = 1$ we have

$$\lim_{n\to\infty} F(z) = \lim_{n\to\infty} f(z) = \infty,$$

so F cannot be analytic there.

(e) If $u(x,y)$ and $v(x,y)$ are harmonic in a domain D then the product $u(x,y)v(x,y)$ is also harmonic in D.

FALSE. Consider $u(x,y) = v(x,y) = x$, clearly harmonic in $\mathbb{R}^2$, but $(uv)(x,y) = x^2$ which has

$$u_{xx} + u_{yy} = 2 \neq 0.$$

2. (i) Find a Möbius transformation T mapping the points $z_0 = 0, z_1 = i+e^{-i\pi/4}, z_2 = i+1$ to $0, 1, \infty$ respectively. (ii) Let D be the intersection of the two open disks $D(1,1) = \{z : |z-1| < 1\}$ $D(i,1) = \{z : |z-i| < 1\}$. Verify that T maps D to the quadrant $Q = \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

Solution: (i) T is given by:

$$T(z) = \frac{z-0}{z-(i+1)} \frac{i+e^{-i\pi/4}-(i+1)}{i+e^{-\pi/4}-0} = \frac{z}{z-(i+1)} \frac{e^{-i\pi/4}-1}{e^{-i\pi/4}+i}.$$

(ii) Observe that the two circles whose arcs form the boundary of D, namely $C_1 : \{|z-1| = 1\}$ and $C_2 : \{|z-i| = 1\}$, intersect at two points: $z_0 = 0$ and $z_2 = i+1$. Since $T(z_0) = 0$ and $T(z_2) = \infty$ the images of these circles are lines L_1 and L_2 through the origin. Since $z_2 \in C_2$ and $T(z_2) = 1 \in \mathbb{R}$, the image of C_2 must be the real line, with $\partial D \cap C_2$ mapped to the ray $(0, \infty)$.

Notice that the tangents to the circles C_1 and C_2, oriented away from the origin, are perpendicular at zero. In particular, the tangent to C_1 at zero makes a positive angle of $\pi/2$ with the tangent to C_2 at zero. Since T is conformal at zero (it is conformal at each point that is not a pole), it preserves angles and we conclude that $T(C_1 \cap \partial D)$ makes an angle of $\pi/2$ with $T(C_2 \cap \partial D)$. Thus, we must have $T(C_1 \cap \partial D) = \{iy : y > 0\}$, the positive imaginary axis, and we conclude that D gets mapped to Q.

3. Verify that $u(x,y) = x^3y - xy^3$ is harmonic in $\mathbb{R}^2$. Find a harmonic conjugate $v(x,y)$ of u and an entire function $f(z)$ such that $f(x+iy) = u(x,y) + iv(x,y)$. Write f as a function of z.

Solution: We have

$$u_{xx} = \frac{d}{dx}\left(3x^2y - y^3\right) = 6xy.$$

$$u_{yy} = \frac{d}{dy}\left(x^3 - 3xy^2\right) = -6xy,$$

so $u_{xx} + u_{yy} = 0$ for all (x, y), so u is harmonic everywhere.

To find a harmonic conjugate v we can solve the Cauchy-Riemann equations (we know that such a conjugate must exist by HW13, since $\mathbb{R}^2$ is simply connected):

$$v_y = u_x = 3x^2y - y^3 \quad \Rightarrow v(x, y) = \int (3x^2y - y^3)dy = 3x^2\frac{y^2}{2} - \frac{y^4}{4} + g_1(x).$$

$$v_x = -(u_y) = -x^3 + 3xy^2 \quad \Rightarrow v(x, y) = \int (3xy^2 - x^3)dx = 3y^2\frac{x^2}{2} - \frac{x^4}{4} + g_2(y).$$

Solving for g_1 and g_2 we find that

$$v(x, y) = \frac{3}{2}x^2y^2 - \frac{x^4}{4} - \frac{y^4}{4}$$

is a harmonic conjugate of $u(x, y)$.

Thus, we have

$$f(x + iy) = (x^3y - yx^3) + i(\frac{3}{2}x^2y^2 - \frac{x^4}{4} - \frac{y^4}{4}).$$

Before we try to figure out what $f(z)$ is, we note that it must be a polynomial of degree 4 in z. For convenience we rewrite:

$$4f(x + iy) = 4(x^3y - yx^3) + i(6x^2y^2 - x^4 - y^4).$$

and recognize these as essentially (upto a factor of i) the binomial coefficients in the expansion:

$$z^4 = (x+iy)^4 = x^4+\binom{4}{1}x^3(iy)+\binom{4}{2}x^2(iy)^2+\binom{4}{1}x(iy)^3+y^4 = x^4+y^4-6x^2y^2+i(4x^3y-4xy^3).$$

Thus, we have $4f(x + iy) = -i(x + iy)^4$ and $f(z) = -iz^4/4$.

4. Consider the polynomial

$$f(z) = z^4 + 5z + 1.$$

How many zeros does f have in the annulus $1 < |z| < 2$?

Solution: Notice that when $|z| = 1$ we have

$$|f(z)| > |5z| - |z^4 + 1| = 5 - 2 > 0$$

so $f(z) \neq 0$ on $|z| = 1$ and $|5z| > |z^4 + 1|$ there. By Rouche's theorem, the number of zeros of f in $|z| < 1$ is equal to the number of zeros of $5z$ there, which is one.

On the other hand, when $|z| = 2$ we have

$$|f(z)| > |z^4| - |5z + 1| = 16 - 11 > 0,$$

so $f(z) \neq 0$ on $|z| = 1$ and $|z^4| > |5z + 1|$ there. By Rouche f must have th same number of zeros in $|z| < 2$ as z^4, which is four.

Subtracting the two, we conclude that f has exactly $4 - 1 = 3$ zeros in the prescribed annulus.

5. Prove that there is no entire function with $\text{Re}(f(z)) = |z|^2$.

   Solution: If such a function existed, its real part would be harmonic, but the function $u(x, y) = x^2 + y^2$ has

$$u_{xx} + u_{yy} = 2 + 2 \neq 0,$$

   so this is impossible.

6. (i) Find the Taylor expansion of $\text{Log}(z)$, the principal branch of the logarithm, centered at $z_0 = 2$. (ii) Consider the branch

$$\text{Log}_\theta(z) = \ln|z| + i\text{Arg}_\theta(z) \quad \text{Arg}_\theta \in (\theta, \theta + 2\pi),$$

   for some $\theta \in (0, \pi)$. What is the radius of convergence of the Taylor series of $\text{Log}_\theta(z)$ at $z_0 = 2$?

   Solution: (i) We have:

$$\begin{aligned} \text{Log}(z) = \text{Log}(2 + (z-2)) \\ &= \text{Log}\,(2 \cdot (1 + (z-2)/2)) \\ &= \text{Log}(1 + \frac{z-2}{2}) + \ln(2) \\ &\quad \text{since multiplication by a real number only changes the magnitude} \\ &= \text{Log}(2) + \frac{z-2}{2} - \frac{(z-2)^2}{3 \cdot 2^2} + \dots. \end{aligned}$$

   Alternatively, letting $f(z) = \text{Log}(z)$, we may compute the derivatives

$$f'(z) = \frac{1}{z} \quad f''(z) = \frac{-1}{z^2} \quad f'''(z) = \frac{2}{z^3} \quad f^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{z^n},$$

   so the Taylor coefficients at $z_0 = 2$ are:

$$a_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n-1}}{n \cdot 2^n},$$

   yielding the Taylor series

$$\text{Log}(z) = \text{Log}(2) + \frac{1}{2}(z-2) - \frac{1}{3 \cdot 2^2}(z-2)^2 + \dots.$$

   (ii) The branch Log_θ has a branch cut along the ray $R_\theta : z = re^{i\theta}$, so the radius of convergence of the Taylor series of $\text{Log}_\theta(z)$ at $z_0 = 2$ is the radius of the largest open disk centered at z_0 disjoint from R_θ. If $\theta \in (\pi/2, \pi)$ the ray lies in the left half plane $\{\text{Re}(z) < 0\}$ and the radius of convergence is the distance to zero, which is 2.

Otherwise, assume $\theta \in (0, \pi/2)$ and let z^* be the point on R_θ closest to z_0 and note that the segment $[z_0, z^*]$ is perpendicular to R_θ. Thus, by considering the triangle with vertices $0, z_0, z^*$ we have

$$\sin(\theta) = \frac{|z_0 - z^*|}{|z_0 - 0|} = \frac{|z_0 - z^*|}{2},$$

so the radius of the circle is $2\sin(\theta)$.

7. Does the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converge *uniformly* in $\mathbb{C}$? Justify your answer.

Solution: No, this series does not converge uniformly. Consider a partial sum $S_N = \sum_{n=0}^{N} \frac{z^n}{n!}$ and let $z > 0$ be a positive real number. The remainder term is then:

$$\rho_N(z) = S(z) - S_N(z) = \sum_{N+1} \frac{z^n}{n!} \geq \frac{z^{N+1}}{(N+1)!}.$$

Taking $z = (N+1)!$ we have $|\rho_N(z)| \geq 1$. Thus, for every N there is a $z \in \mathbb{C}$ such that $|\rho_N(z)| \geq 1$, so the series cannot converge uniformly.

8. Prove the Casorati-Weierstrass theorem: if $f(z)$ has an essential singularity at z_0 then for every $w_0, \epsilon > 0$, and $\delta > 0$, there exists $|z - z_0| < \delta$ such that $|f(z) - w_0| < \epsilon$. (you can use any theorem on classification of singularities.)

Solution: See page 257 of the textbook.

9. Evaluate the integral

$$\int_0^{\infty} \frac{\sin(x)}{x(x^2+1)} dx,$$

where we understand the integrand as extending to a continuous function with value $\lim_{x\to 0} \sin(x)/(x(x^2+1))$ at $x = 0$.

Solution: Since the integrand is even, we may write

$$2I = \text{p.v.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x(x^2+1)} dx = \text{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z(z^2+1)} dz = \text{Im} \lim_{\rho\to 0, R\to\infty} \int_{-R}^{-\rho} f(z)dz + \int_{\rho}^{R} f(z)dz,$$

where we have taken

$$f(z) := \frac{e^{iz}}{z(z^2+1)} dz.$$

The integrand has simple poles at $z = 0$ and $\pm i$, so we use an indented contour which avoids the pole at zero. Let $C_1 : z(t) = z, t \in [-R, -\rho]$ and $C_2 : z(t) = z, t \in [\rho, R]$ be contours along the intervals above, and consider the semicircular contours:

$$C_\rho : z(t) = \rho e^{it}, t \in [0, \pi] \quad C_R : z(t) = Re^{it}, t \in [0, \pi].$$

We then have for every $\rho, R > 0$:

$$\operatorname{Im} \oint_{C_1 - C_\rho + C_2 + C_R} f(z)dz = 2I - \operatorname{Im} \int_{C_\rho} f(z)dz + \operatorname{Im} \int_{C_R} f(z)dz.$$

We evaluate the integral on the left hand side using the residue theorem. For small ρ and large R, the only singularity of

$$f(z) = \frac{e^{iz}}{z(z+i)(z-i)}$$

inside $C = C_1 - C_\rho + C_2 + C_R$ is at $z = i$. The residue there is:

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{e^{iz}}{z(z+i)} = -\frac{e^{-1}}{2},$$

so

$$\operatorname{Im} \oint_C f(z)dz = \operatorname{Im}(2\pi i \cdot \frac{e^{-1}}{-2}) = -\frac{\pi}{e}.$$

On the right hand side, we observe that since $|e^{iz}| \leq 1$ in the upper half plane, an ML estimate tells us that

$$\left| \oint_{C_R} \frac{e^{iz}}{z(z^2+1)} dz \right| \leq \pi R \cdot \frac{1}{R(R^2-1)} \to 0$$

as $R \to \infty$.

The other integral picks up $-i\pi$ times the residue at zero (since it is oriented clockwise) as $\rho \to 0$; this residue is

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} \frac{e^{iz}}{z^2+1} = 1,$$

so we have

$$\operatorname{Im} \lim_{\rho \to 0} \int_{C_\rho} f(z)dz = \operatorname{Im}(-i\pi \cdot 1) = -\pi.$$

Combining these results, we have:

$$\frac{-\pi}{e} = 2I - \pi \quad \Rightarrow \quad I = \pi(1 - e^{-1})/2.$$