Math 185 Fall 2015, Sample Final Exam

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Questions 1, 8, 9 are worth 15 points, and Question 5 is worth 5 points. The rest are worth 10 points.

1. True or false:

   (a) If f is analytic in the annulus $A = \{z : 1 < |z| < 2\}$ then there exist functions g and h such that g is analytic in $|z| < 2$, h is analytic in $|z| > 1$, and $f = g + h$ in A

   (b) There is a simple closed contour in $\mathbb{C} \setminus \{0\}$ such that

   $$\oint_C \frac{1}{z^2} dz = 2\pi i$$

   .

   (c) If $f(z)$ and $g(z)$ have poles at z_0 then $\mathrm{Res}(fg, z_0) = \mathrm{Res}(f, z_0)\mathrm{Res}(g, z_0)$.

   (d) Every function analytic in $D(0, 1)$ has an analytic continuation to $D(0, 2)$.

   (e) If $u(x, y)$ and $v(x, y)$ are harmonic in a domain D then the product $u(x, y)v(x, y)$ is also harmonic in D.

2. (i) Find a Möbius transformation T mapping the points $z_0 = 0$, $z_1 = i+e^{-i\pi/4}$, $z_2 = i+1$ to $0, 1, \infty$ respectively. (ii) Let D be the intersection of the two open disks $D(1, 1) = \{z : |z - 1| < 1\}$ $D(i, 1) = \{z : |z - i| < 1\}$. Show that T maps D to the quadrant $Q = \{z : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$.

3. Verify that $u(x, y) = x^3y - xy^3$ is harmonic in $\mathbb{C}$. Find a harmonic conjugate $v(x, y)$ of u and an entire function $f(z)$ such that $f(x + iy) = u(x, y) + iv(x, y)$. Write f as a function of z.

4. Consider the polynomial

   $$f(z) = z^4 + 5z + 1.$$

   How many zeros does f have in the annulus $1 < |z| < 2$?

5. Prove that there is no entire function with $\mathrm{Re}(f(z)) = |z|^2$.

6. (i) Find the Taylor expansion of $\text{Log}(z)$, the principal branch of the logarithm, centered at $z_0 = 2$. (ii) Consider the branch

$$\text{Log}_\theta(z) = \ln|z| + i\text{Arg}_\theta(z) \quad \text{Arg}_\theta \in (\theta, \theta + 2\pi),$$

for some $\theta \in (0, \pi)$. What is the radius of convergence of the Taylor series of $\text{Log}_\theta(z)$ at $z_0 = 2$?

7. Does the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converge *uniformly* in $\mathbb{C}$? Justify your answer.

8. Prove the Casorati-Weierstrass theorem: if $f(z)$ has an essential singularity at z_0 then for every $w_0, \epsilon > 0$, and $\delta > 0$, there exists a z with $0 < |z - z_0| < \delta$ such that $|f(z) - w_0| < \epsilon$. (you can use any theorem on classification of singularities.)

9. Evaluate the integral

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)} dx,$$

where we understand the integrand as extending to a continuous function with value $\lim_{x\to 0} \sin(x)/(x(x^2+1))$ at $x = 0$.