Math 185 Fall 2015, Sample Final Exam

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Questions 1, 8, 9 are worth 15 points, and Question 5 is worth 5 points. The rest are worth 10 points.

1. True or false:
   (a) If \( f \) is analytic in the annulus \( A = \{ z : 1 < |z| < 2 \} \) then there exist functions \( g \) and \( h \) such that \( g \) is analytic in \( |z| < 2 \), \( h \) is analytic in \( |z| > 1 \), and \( f = g + h \) in \( A \).
   (b) There is a simple closed contour in \( \mathbb{C} \setminus \{0\} \) such that
       \[ \oint_C \frac{1}{z^2} \, dz = 2\pi i. \]
   (c) If \( f(z) \) and \( g(z) \) have poles at \( z_0 \) then \( \text{Res}(fg, z_0) = \text{Res}(f, z_0) \text{Res}(g, z_0) \).
   (d) Every function analytic in \( D(0,1) \) has an analytic continuation to \( D(0,2) \).
   (e) If \( u(x, y) \) and \( v(x, y) \) are harmonic in a domain \( D \) then the product \( u(x, y)v(x, y) \) is also harmonic in \( D \).

2. (i) Find a Möbius transformation \( T \) mapping the points \( z_0 = 0, z_1 = i + e^{-i\pi/4}, z_2 = i + 1 \) to \( 0, 1, \infty \) respectively. (ii) Let \( D \) be the intersection of the two open disks \( D(1,1) = \{ z : |z - 1| < 1 \} \) \( D(i,1) = \{ z : |z - i| < 1 \} \). Show that \( T \) maps \( D \) to the quadrant \( Q = \{ z : \text{Re}(z) > 0, \text{Im}(z) > 0 \} \).

3. Verify that \( u(x, y) = x^3y - xy^3 \) is harmonic in \( \mathbb{C} \). Find a harmonic conjugate \( v(x, y) \) of \( u \) and an entire function \( f(z) \) such that \( f(x + iy) = u(x, y) + iv(x, y) \). Write \( f \) as a function of \( z \).

4. Consider the polynomial
   \[ f(z) = z^4 + 5z + 1. \]
   How many zeros does \( f \) have in the annulus \( 1 < |z| < 2 \)?

5. Prove that there is no entire function with \( \text{Re}(f(z)) = |z|^2 \).
6. (i) Find the Taylor expansion of \( \log(z) \), the principal branch of the logarithm, centered at \( z_0 = 2 \). (ii) Consider the branch
\[
\log_\theta(z) = \ln |z| + i \text{Arg}_\theta(z) \quad \text{Arg}_\theta \in (\theta, \theta + 2\pi),
\]
for some \( \theta \in (0, \pi) \). What is the radius of convergence of the Taylor series of \( \log_\theta(z) \) at \( z_0 = 2 \)?

7. Does the series
\[
\sum_{n=0}^{\infty} \frac{z^n}{n!}
\]
converge uniformly in \( \mathbb{C} \)? Justify your answer.

8. Prove the Casorati-Weierstrass theorem: if \( f(z) \) has an essential singularity at \( z_0 \) then for every \( w_0, \epsilon > 0 \), and \( \delta > 0 \), there exists a \( z \) with \( 0 < |z - z_0| < \delta \) such that \( |f(z) - w_0| < \epsilon \). (you can use any theorem on classification of singularities.)

9. Evaluate the integral
\[
\int_0^\infty \frac{\sin(x)}{x(x^2 + 1)} dx,
\]
where we understand the integrand as extending to a continuous function with value \( \lim_{x \to 0} \frac{\sin(x)}{x(x^2 + 1)} \) at \( x = 0 \).