Math 185 Fall 2015, Final Exam

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8:10am–11:00am, December 17, 2015, 9 Evans Hall

1. (12 points) True or False (no need for justification):

   (a) If $f(z)$ is analytic and bounded in the right half plane $H = \{z : \text{Re}(z) > 0\}$ then f must be constant in H.

   (b) If $u(x, y)$ is harmonic in $\mathbb{R}^2$ and there is an $M > 0$ such that $|u(x, y)| \leq M$ for all x, y then u must be constant.

   (c) $f(z) = e^z$ is one to one in the unit disk $D(0, 1)$.

   (d) If $f(z)$ has a pole of order m at $z_0 = 0$ then $f(z^2)$ has a pole of order $2m$ at $z_0 = 0$.

   (e) If $f(z)$ has a removable singularity at z_0 then $\text{Res}(f, z_0) = 0$.

   (f) If f is analytic in $D(0, 1)$ and $|f(z)| \leq 1$ for all $z \in D(0, 1)$ then $|f(z)| < 1$ for all $z \in D(0, 1)$.

2. (10 points) Suppose $u(x, y)$ and $v(x, y)$ are harmonic in a domain D and v is the harmonic conjugate of u. (i) Prove that $u^2 - v^2$ is harmonic in D. (ii) Prove that the partial derivative u_x is harmonic in D.

3. (10 points) (i) Find a Möbius transformation which maps the open half-disk

   $$S = \{z : |z| < 1, \text{Im}(z) > 0\}$$

   to a quadrant. (ii) Find a 1-1 conformal mapping of the quadrant to the upper half-plane. (iii) Find a Möbius transformation mapping the upper halfplane to the unit disk $D = \{z : |z| < 1\}$. (iv) Compose these to give a 1-1 conformal map of the half-disk to the unit disk.

   Justify each step (i.e., explain why the transformations you produce have the desired properties).

   What goes wrong if you try to directly use the transformation $w = z^2$ to map S conformally and 1-1 onto D?

4. (9 points) Classify (as removable, pole, or essential) the singularity at $z = 0$ of the following functions, and explain why. If it is a pole calculate the residue.

$$\frac{\operatorname{Log}(z+1)\sin(z)}{z^2} \qquad e^{\sin(1/z)} \qquad \frac{1+z}{e^z-1},$$

where Log is the principal branch.

5. (10 points) Show that there exists an $\epsilon > 0$ such that for every polynomial $p(z)$:

$$\max_{|z|=1}\left|\frac{1}{z} - p(z)\right| > \epsilon.$$

(hint: argue by contradiction.)

6. (10 points) Prove or disprove: there is a function f analytic in $D(0,1)$ with the property that

$$f\left(\frac{1}{n^2}\right) = \frac{1}{n^3},$$

for all integers $n > 1$.

7. (7 points) Suppose f is analytic in the closed disk $\overline{D}(0,1) = \{z : |z| \leq 1\}$ and $|f(z)| < 1$ for all $z \in \overline{D}(0,1)$. Prove that f has a unique fixed point in $D(0,1)$ (i.e., there is a unique z_0 such that $f(z_0) = z_0$).

8. (12 points) State and prove the Cauchy Integral Formula. (you may assume the Cauchy-Goursat theorem).

9. (10 points) Evaluate the integral:

$$\int_0^{\pi} \frac{1}{2+\sin(2\theta)}\,d\theta.$$

Justify all steps.

10. (10 points) Evaluate the integral:

$$\int_0^{\infty} \frac{\cos(2x)-1}{x^2}\,dx.$$

Justify all steps.

Good luck, and have a good break!