Analytic Continuation and Γ

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1 Analytic Continuation

Analytic continuation means extending an analytic function defined in a domain to one defined in a larger domain.

Definition 1.1. If $f(z)$ is analytic in a domain D and $F(z)$ is analytic in a domain $D' \supset D$ with $F(z) = f(z)$ in D, then we say that F is an *analytic continuation* of f.

It is used as both a noun (above) and as a verb ("to analytically continue"), to indicate the process of defining the extension. The simplest example is given by the geometric series at zero:

$$f(z) = 1 + z + z^2 + \dots,$$

which converges in the open disk $D = \{|z| < 1\}$ and defines an analytic function there. If we multiply by z and subtract we obtain the familiar formula:

$$zf(z) - f(z) = 1 \quad \Rightarrow f(z) = \frac{1}{1-z} \quad \Rightarrow f(z) = \frac{1}{1-z} \quad z \in D,$$

which is a *functional equation* satisfied by $f(z)$. The crucial point is that the expression on the right hand side makes sense for a much larger set of z, in particular it defines an analytic function $F(z) = \frac{1}{1-z}$ in the larger domain $D' = \mathbb{C} \setminus \{1\}$. Thus, F is an analytic continuation of f.

The general principle is that there are many ways of describing a function, and some descriptions make sense in larger regions than others. In particular, sometimes the description we start off with (such as the power series above) is often only a small piece of a much larger picture, which is revealed by the analytic continuation.

The reason it makes sense to speak of *the* analytic continuation is the following uniqueness property, which is an immediate consequence of the identity theorem.

Theorem 1.2. *If $F_1 : D' \to \mathbb{C}$ and $F_2 : D' \to \mathbb{C}$ are two analytic continuations of $f : D \to \mathbb{C}$ then $F_1(z) = F_2(z)$ for all $z \in D'$.*

Proof. Notice that $F_1(z) - F_2(z)$ vanishes on $D \subset D'$, which is an open set. By the identity theorem F_1 and F_2 must be the same. □

Remark 1.3. The uniqueness property requires the domains of the two analytic continuations to be the same. It is *not* generally true that if $F_1 : D_1 \to \mathbb{C}$ and $F_2 : D_1 \to \mathbb{C}$ are two analytic continuations of $f : D \to C$ to different domains D_1, D_2, that they must agree on $D_1 \cap D_2$.

A slightly more complicated example is the power series with Fibonacci coefficients:

$$f(z) = f_0 + f_1 z + f_2 z^2 + \ldots,$$

which we considered a few lectures ago. Initially we observed that this converges and thereby defines an analytic function in some neighborhood D of zero. By applying the recurrence $f_{n+1} = f_n + f_{n-1}$, we were able to obtain the functional equation:

$$(1 - z - z^2)f(z) = z \quad \Rightarrow f(z) = \frac{z}{1 - z - z^2} \quad z \in D.$$

We then used the right hand side as a *definition* of f in a much larger domain $D' = \mathbb{C}\backslash\{\phi, \psi\}$. Formally, $F(z) = \frac{z}{1-z-z^2}$ is an analytic continuation of f to D'. We didn't explicitly use a different name to distinguish between the continuation and the original function (since they agree where they are both defined) and we will sometimes follow this convention in the future. In any case, we were then able to use the properties of F in the much larger domain D' (by applying the Residue theorem) to get a good handle on what is happening at zero, and thereby extract a formula for the coefficients.

A functional equation is not the only way to obtain an analytic equation, but it is often the best one. In general, what one is looking for is an *alternate representation* of the same function which makes sense in a larger region; this alternate description is then used as a *definition* in the larger region. The ones we obtained above were closed form formulas. Below we look at a more interesting example where this is not the case.

2 The Gamma Function

The Gamma Function is a meromorphic function which extends the factorial function to all complex numbers other than the nonpositive integers. It is defined as an integral:

Definition 2.1.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

We will verify in a moment that it indeed matches the factorial, but first let us figure out where this integral converges and where the definition makes sense. Observe that it converges in the upper limit for real $z > 0$ because e^{-t} decays faster than any power t^{z-1} grows; in fact it converges uniformly in any interval $z \leq a$ since $t^{z-1} \leq t^{a-1}$ in such an interval. It also converges in the lower limit since $t^{z-1} = 1/t^{1-z}$ with $1 - z < 1$ and $e^{-t} \leq 1$ in every neighborhood of 1; moreover it converges uniformly in every set $z \geq b$ since $t^{z-1} \leq t^{b-1}$ for $|t| < 1$. All in all, the integral converges uniformly in every interval $[b, a]$ with $b > 0$.

To handle the case of complex z, we observe that

$$\left|\int_0^\infty e^{-t}t^{z-1}dt\right| \leq \int_0^\infty |e^{-t}t^{z-1}|dt = \int_0^\infty e^{-t}t^{\mathrm{Re}(z)-1}dt$$

converges in $\mathrm{Re}(z) > 0$ and uniformly in every strip $0 < b \leq \mathrm{Re}(z) \leq a$. Thus, the function is well-defined for $\mathrm{Re}(z) > 0$. To see that it is analytic there, we observe that any point in $\mathrm{Re}(z) > 0$ is contained in a strip of this type, so uniform convergence allows us to differentiate under the integral sign (by going to the definition of the derivative and applying an ML estimate type of argument), and we get

$$\frac{d}{dz}\Gamma(z) = \int_0^\infty \frac{d}{dz}e^{-t}t^{z-1}dt = \int_0^\infty e^{-t}t^{z-1}\log(t)dt,$$

so $\Gamma(z)$ is analytic when $\mathrm{Re}(z) > 0$.

Let us now plug in some values to see the relationship with the factorial. We evaluate

$$\Gamma(1) = 1 \quad \Gamma(2) = 1 \quad \Gamma(3) = 2\ldots.$$

The general idea is to use integration by parts to get $\Gamma(z+1)$ from $\Gamma(z)$, for $\mathrm{Re}(z) > 0$:

$$\Gamma(z+1) = \int_0^\infty e^{-t}t^z dt = t^z(-e^{-t})_0^\infty - \int_0^\infty zt^{z-1}(-e^{-t})dt = 0 + z\Gamma(z).$$

Thus, we have the recursive functional equation:

$$\Gamma(z+1) = z\Gamma(z).$$

In particular, at the integers we obtain $\Gamma(n) = (n-1)\Gamma(n-1) = \ldots = (n-1)!$.

We are now going to use the functional equation to analytically continue $\Gamma(z)$ to a larger region. The equation tells us how to get $z+1$ from z but it also tells us how to get z from $z+1$:

$$\Gamma(z) = \Gamma(z+1)/z.$$

We can use this as a *definition* to obtain an extension $\Gamma_1 : D_1 \to \mathbb{C}$ where $D_1 = \{\mathrm{Re}(z) > -1\}$ as

$$\Gamma_1(z) = \Gamma(z+1)/z$$

when $z \in D_1 \setminus D_0$ and just $\Gamma(z)$ otherwise. Note that this only works when $z \neq 0$ and so it has a pole there. THe punch line is that this function is automatically analytic at $z \in D_1$ since $z+1 \in D_0$ and Γ is analytic at $z+1$. Since it is an analytic function which agrees with Γ on D_0 it must be the unique analytic continuation to D_1.

There is no reason to stop here: by doing the same thing again we can obtain a continuation

$$\Gamma_2(z) = \Gamma(z+2)/z(z+1).$$

If we do this m times we obtain an analytic function in $D_m = \{\mathrm{Re}(z) > -m\}$:

$$\Gamma_m(z) = \Gamma(z+m)/z(z+1)\ldots(z+m-1),$$

with poles at $0, -1, \ldots -(m-1)$. In general we can define an function $\Gamma(z)$ analytic in $\mathbb{C} \setminus \{0, -1, \ldots\}$, which we refer to as "the" Gamma function.

This will come up again in the prime number theorem, but it comes up in a variety of other contexts.

3 The PNT and the ζ function

A number $n \neq 1$ is prime if has no divisors other than 1 and itself. The fundamental theorem of arithmetic says that every integer has a unique factorization as a product of prime powers. Euclid proved that there are infinitely many primes in 300BC. However, you can ask: how big of an infinity? Or even, how likely is a 'random' number to be prime? It is not easy to get a handle on the distribution of prime numbers. For instance, what is the number of primes less than 1000?

The Prime Number Theorem, conjectured by Gauss and Legendre, provides a sharp answer to this question. It is too much to expect to get an exact formula like $\sqrt{x}$, but we can get close. Let $\pi(x)$ denote the number of primes less than $x > 1$. Then

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \log(x)} = 1.$$

The Riemann Zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is easy to see that it converges for $\mathrm{Re}(s) > 1$, and earlier we calculated $\zeta(2)$ and $\zeta(4)$. The value of $\zeta(3)$ is still open. The reason it is interesting in this context is the Euler Product Formula:

$$\zeta(s) = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1} \quad \mathrm{Re}(s) > 1,$$

which connects the values of zeta to the distribution of primes, much like in the coin exchange problem on HW10. In particular, the solution to the Basel problem $\pi^2/6$ already implies that there are infinitely many primes.

Currently ζ is only defined right of the line $\mathrm{Re}(s) = 1$, but it turns out it has a whole other life left of this line, which is where we will look to understand the distribution of the primes.