Supplementary Notes for Week 13

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Roughly we covered Sections 96-103 and 114-117 of Brown and Churchill, but my presentation of these topics was a bit different from what is in the book. If you missed any of the lectures on Möbius transformations I suggest also reading section II.7 of Gamelin.

1 The Inverse Function Theorem

Theorem. Suppose $f'(z_0) \neq 0$. Then there is a neighborhood of z_0 in which f is 1-1.

Proof. Since $f'(z)$ is continuous at z_0, we can choose $r > 0$ such that $|f'(z) - f'(z_0)| < |f'(z_0)|/2$ whenever $z \in D(z_0, r)$. Let z_1, z_2 be two distinct points in $D(z_0, r)$, and observe that

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)dz,$$

where the integral is taken along the line segment between z_1 and z_2, which is also contained in $D(z_0, r)$. We then have

$$\begin{aligned}
|f(z_2) - f(z_1)| &= \left| \int_{z_1}^{z_2} f'(z_0)dz + \int_{z_1}^{z_2} (f'(z) - f'(z_0))dz \right| \\
&\geq \left| \int_{z_1}^{z_2} f'(z_0)dz \right| - \left| \int_{z_1}^{z_2} (f'(z) - f'(z_0))dz \right| \quad \text{triangle inequality} \\
&\geq |f'(z_0)||z_2 - z_1| - \max_{z \in [z_1, z_2]} |f'(z) - f'(z_0)| \cdot |z_2 - z_1| \quad \text{ML estimate} \\
&\geq |f'(z_0)||z_2 - z_1| - \frac{|f'(z_0)|}{2}|z_2 - z_1| \quad \text{since } z_1, z_2 \in D(z_0, r) \\
&\geq \frac{|f'(z_0)|}{2}|z_2 - z_1|.
\end{aligned}$$

In particular, if $z_1 \neq z_2$ then $f(z_1) \neq f(z_2)$. □

The rest of the theorem (that there is an inverse function defined on a neighborhood of $f(z_0)$) is in HW13.

2 The Dirichlet Problem in the Upper Halfplane

There is a rather easy way to solve any boundary value problem which asks for a function $h(x, y)$ harmonic in the upper half plane satisfying boundary conditions of type:

$$h(x,0) = \begin{cases} \alpha_0 & x \in (-\infty, z_0) \\ \alpha_1 & x \in (z_0, z_1) \\ \dots & \\ \alpha_i & x \in (z_{i-1}, z_i) \\ \dots & \\ \alpha_{n+1} & x \in (z_n, \infty) \end{cases}.$$

for finitely many real points $z_0 < z_1 < \ldots < z_n$ and boundary values $\alpha_0, \ldots, \alpha_{n+1} \in \mathbb{R}$. This is based on two observations:

1. It is easy to solve the problem when $n = 0$: we observe that the function $\mathrm{Arg}(z - z_0)$, where $\mathrm{Arg}(z) \in (-\pi/2, 3\pi/2)$, has value 0 when $z > z_0$ and π when $z < z_0$. Moreover $\mathrm{Arg}(z)$ is harmonic since it is the imaginary part of a branch of $\mathrm{Log}(z)$ with a branch cut on the negative imaginary axis.

   We can achieve any boundary values α_0, α_1 (which may be different from $\pi, 0$) by considering an affine transformation $A \cdot \mathrm{Arg}(z - z_0) + B$ and solving for A and B by plugging in points on both sides:
   $$A \cdot \pi + B = \alpha_0 \qquad A \cdot 0 + B = \alpha_1.$$

   This system always has a solution since it triangular.

2. Since linear combinations of harmonic functions are harmonic, we can try to solve the general problem by considering a sum of solutions to the $n = 0$ case. In particular, we can consider a solution of type

   $$h(z) = A_0 \cdot \mathrm{Arg}(z - z_0) + \ldots A_n \cdot \mathrm{Arg}(z - z_n) + B,$$

   and try to fit it to the boundary values by plugging in values in the intervals (z_{i-1}, z_n) (observing again that for any interval (z_{i-1}, z_i) we have $\mathrm{Arg}(z - z_i) = \pi$ whenever $z < z_{i-1}$ and $\mathrm{Arg}(z - z_i) = 0$ whenever $z > z_i$). Since there are $n + 1$ variables and $n + 1$ constraints and the linear system is again triangular, it always has a solution. (In concrete terms, what this means is that the linear equation corresponding to the rightmost interval will only have one variable, the next one will have two variables, etc.)

By applying appropriate Möbius transformations this method can be used to obtain solutions to Dirichlet problems on other domains.