Solving Recurrences

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In this lecture we will see how to use complex analysis to solve an a priori discrete problem. The Fibonacci numbers are an infinite sequence defined implicitly by the recursion relation:

$$f_0 = 0 \quad f_1 = 1 \quad f_2 = 1 \quad \ldots f_{n+2} = f_{n+1} + f_n \ldots.$$

This definition explains the process that generates the numbers – adding the two previous numbers – but does not give an explicit formula for the nth number (it is unclear that one exists) or a clear idea of, for instance, how fast these numbers grow.

We will be able to obtain both by finding a *generating function* for the f_n. The first step is to consider the power series

$$F(z) := \sum_{n=0}^{\infty} f_n z^n$$

defined by the f_n. Note that $F(0) = 0$ and $F(z)$ converges in a neighborhood of 0 (for instance, by observing that $f_{n+1} \leq 2f_n$, so that $f_n \leq 2^n$ and applying a comparison test to the geometric series), so F is analytic at zero. The numbers we are interested in are (upto factorials) the coefficients of the Taylor series of F at zero, which may also be written as:

$$f_n = \frac{F^{(n)}(0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{F(z)}{z^{n+1}}dz = \mathrm{Res}(\frac{F(z)}{z^{n+1}}, 0),$$

by applying the Cauchy Integral Formula and the Residue theorem. We will now calculate these residues in a different way.

The key is to use the recursion relation to find a *functional equation* satisfied by the $F(z)$. In this case, we know that f_{n+2} as the sum of the two terms before it, so there should be a similar relationship between $F(z)$ and $zF(z)$ and $z^2F(z)$, which are just $F(z)$ with coefficients shifted by one and two places. A moment's thought reveals that after throwing away the first two coefficients, we simply have

$$F(z) - f_1 z - f_0 = F(z) - z = zF(z) + z^2 F(z),$$

which after rearranging is

$$F(z) = \frac{z}{1 - z - z^2}.$$

This is a different (and much more powerful way) of representing the series f_n. In particular, note that $F(z)$ is a rational function that is analytic everywhere except for the zeros of $1 - z - z^2$. A simple calculation shows that these zeros are given by $z = \frac{-1\pm\sqrt{5}}{2}$, which we will refer to as ϕ and ψ (the first is just minus the golden ratio). Note that $\phi\psi = -1$ and $\phi + \psi = -1$.

We now consider the function $F(z)/z^{n+1}$, whose residues at zero are the Fibonacci numbers. This function has a pole of order n at zero and simple poles at ϕ, ψ. Let C_R be a circle of radius R centered at zero containing all of these poles. Observe that $F(z)/z^{n+1} = \frac{-1}{z^n(z^2+z-1)}$ is the reciprocal of a polynomial of degree at least two. By the midterm extra credit problem, this means that

$$0 = \frac{1}{2\pi i}\oint_{C_R} F/z^{n+1}dz = \text{Res}(F(z)/z^{n+1}, 0) + \text{Res}(F(z)/z^{n+1}, \phi) + \text{Res}(F(z)/z^{n+1}, \psi),$$

by the Residue theorem. Rearranging and easily computing the residues at the simple poles ϕ and ψ yields the formula

$$f_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}.$$

This is the simplest instance of the use of generating functions that I can think of. This method is extremely versatile and there is a whole field, known as *analytic combinatorics* devoted to it. The idea is that the behavior of various (implicitly defined) combinatorial sequences of numbers can be precisely understood by studying the singularities of the associated generating functions. This comes up a lot, for instance, in the analysis of algorithms.

See the books "generatingfunctionology" by Wilf or "Analytic Combinatorics" by Fajolet and Sedgewick for more details.