The Basel Problem, The Point at Infinity

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1 The Basel Problem

In class I proved that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$ using the Residue theorem. This was at some point a very famous open problem known as the "Basel Problem", and it was solved in 1728 by Euler, though his proof was not entirely rigorous.

We will prove this using the residue theorem. The idea is to consider integrals of the function

$$f(z) = z^{-2}\pi \cot(\pi z)$$

whose residues at the integers are equal to $1/n^2$, on contours B_N which were axis-aligned squares with sides passing through $\pm N/2, \pm iN/2$ where N is an odd integer. The reason this paid off is that one can show that $f(z) = O(1/N^2)$ on B_N, which follows from the claim that

$$g(z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

is bounded on B_N for large odd N.

Here is the complete proof of that claim. Observe that

$$|g(z)| = \frac{|e^{i\pi z} + e^{-i\pi z}|}{|e^{i\pi z} - e^{-i\pi z}|} = \frac{|e^{i\pi 2x}e^{-\pi 2y} + 1|}{|e^{i\pi 2x}e^{-\pi 2y} - 1|}$$

for $z = x + iy$. This quantity is small for different reasons on the different sides of B_N. When $y = N/2$ we have $e^{-\pi y} = e^{-\pi N/2} < e^{-3}$ which is small

$$|g(z)| \leq \frac{1 + e^{-\pi N/2}}{1 - e^{-\pi N/2}} < 2.$$

When $y = -N/2$ a similar argument shows that the $e^{\pi N/2}$ term dominates. When $x = iN/2$ (and this is where the oddness is crucial), we observe that

$$e^{i\pi 2x} = e^{i\pi N}$$

is negative, so

$$|g(z)| = \frac{|-e^{-\pi 2y} + 1|}{|-1 - e^{-\pi 2y}|} < 1,$$

and a similar argument holds for $x = -iN/2$. In all cases we have $|g(z)| < 2$ on B_N.

Given the claim, the proof is completed by applying the Residue theorem and noting that for every N:

$$2\sum_{n=1}^{(N-1)/2} \text{Res}(f, n) + \text{Res}(f, 0) = \frac{1}{2\pi i}\oint_{B_N} f(z)dz$$

and letting $N \to \infty$. The limit of the partial sums on the left is $\sum_{n=1}^{\infty} 1/n^2$, and explcit computation reveals that the residue at zero is $\pi^2/3$.

This method can be used to sum any series $\sum_{n=1}^{\infty} f(n)$ where $f(-n) = f(n)$ and f is analytic except at zero, by considering the function $f(z)\pi\cot(\pi z)$. Variants of the method can be used to sum other series.

The difference between this and other contour integration methods we have seen is that we carefully chose the contours B_N to "snap to a grid" where the integrand is bounded.

2 The Point at Infinity

Let $\hat{\mathbb{C}}$ denote the extended complex plane $\mathbb{C} \cup \{\infty\}$. We think of $\hat{\mathbb{C}}$ as the Riemann sphere. Recall that a set of the form $\{z \in \mathbb{C} : |z| > R\}$ is called a neighborhood of ∞.

We say that $f(z)$ is analytic/has a removable singularity/pole/essential singularity at ∞ if $f(1/z)$ is analytic/has a removable singularity/pole/essential singularity (respectively) at 0.

For instance $f(z) = z$ has a simple pole at infinity, $f(z) = e^z$ has an essential singularity at ∞, and $f(z) = \frac{1}{z^2}$ is analytic at infinity. All the theorems we proved concerning the limiting behavior of functions near isolated singularities (such as Riemann's theorem on isolated singularities and the Casorati-Weierstrass theorem; see Section 84) continue to apply when there is an isolated singularity at ∞.

There is a very pleasing classification of functions which are meromorphic in $\mathbb{C}$ and have anything less severe than an essential singularity at ∞. This is outlined in the following three theorems.

Theorem 1. If f is analytic in $\hat{\mathbb{C}}$ then f is constant.

Proof. This means that $f(1/z)$ is analytic at zero, so $\lim_{z\to 0} f(1/z) = L$ for some finite L. This means that there is an $\epsilon > 0$ such that $|f(1/z)| < 2L$ whenever $|z| < \epsilon$ which is the same as saying that $|f(z)| < 2L$ whenever $|z| > 1/\epsilon$, i.e., f is bounded in a neighborhood neighborhood of ∞. But then $f(z)$ is bounded in the whole complex plane (since it is also bounded in the closed disk $\{z : |z| \leq 1/\epsilon\}$ because f is continuous), so by Liouville's theorem it is constant. □

Theorem 2. If f is analytic in $\mathbb{C}$ and has a pole at ∞ then f is a polynomial.

I tried to give an analytic proof of this theorem and got stuck, but here is a much slicker proof using the series representation due to David (and perhaps also what Praagya was thinking):

Proof. Since f is entire it has a Taylor series at the origin which converges everywhere: $f(z) = \sum_{n=0}^{\infty} a_n z_n$, $|z| < \infty$. This is equivalent to saying that the Laurent series

$$f(1/z) = \sum_{n=0} \frac{a_n}{z^n}$$

converges in $0 < |z| < \infty$. But since $f(1/z)$ has a pole at 0 the principal part can have only finitely many terms (otherwise it would be an essential singularity of $f(1/z)$ at zero, which is an essential singularity of $f(z)$ at ∞), so only finitely many a_n are nonzero, and f is a polynomial. □

Theorem 3. If f is meromorphic in $\hat{\mathbb{C}}$ then f is a rational function.

Proof. Since f is meromorphic the only kind of singularities it can have are poles (if it has removable singularities, redefine the function at points to remove them). Observe that there must be some radius R such that all of the poles of $f(z)$ are contained in the disk $\{|z| \leq R\}$; otherwise, there would be a pole in every punctured neighborhood of ∞, and ∞ would not be an isolated singularity, which is a contradiction.

Let $z_1, \ldots, z_k$ be the poles of $f(z)$. Note that $f(z)$ has a Laurent series at each z_i with a principal part containing finitely many terms; let $g_1(z), \ldots, g_k(z)$ be these principal parts and note that each g_i is a rational function[1] Now observe that the function

$$h(z) := f(z) - \sum_{i=1}^{k} g(z)$$

is analytic in $\mathbb{C}$ except for removable singularities at $z_1, \ldots, z_k$. Moreover, it has a pole at ∞ because $f(z)$ has a pole at ∞ and $\lim_{z\to\infty} g_i(z) = 0$ for all i. Remove these singularities by redefining $h(z_i) = \lim_{z\to z_i} h(z)$. Now $h(z)$ is analytic except for a pole at ∞, so the previous theorem implies that $h(z)$ must be a polynomial. But now $f(z)$ is a rational function, as desired. □

3 The Residue at Infinity

This is covered in Section 77 of the book. The residue of $f(z)$ at infinity is defined as an integral

$$\frac{1}{2\pi i} \oint_C f(z) dz$$

where C is any *negatively* oriented simple closed contour such that f is analytic in the exterior of C.

[1]For concreteness, one may write each $g_i(z) = \sum_{n=1}^{m_i} \frac{b_{n,i}}{(z-z_i)^n}$ for some coefficients $b_{n,i}$ depending on i, where m_i is the order of the pole z_i.

The idea is that if f has finitely many poles, it must have a Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$$

about zero that is convergent in a punctured neighborhood of ∞, i.e., in a set of the form $R < |z| < \infty$ (an annulus). By the deformation theorem we can assume that C is contained in this annulus, and we we are interested in the coefficient c_{-1} of $1/z$, for which we have $-c_{-1} = \operatorname{Res}(f, \infty)$, with the minus sign appearing because of the change in orientation.

The upshot is that by making a change of variables, the Laurent series may equivalently be written as:

$$f(1/z) = \sum_{n\in Z} \frac{c_n}{z^n}$$

is convergent in $R < |1/z| < \infty$ i.e. in $0 < |z| < 1/R$, which is a neighborhood of zero. Note that this is the original Laurent series with the coefficients reversed. However, the residue of $f(1/z)$ at zero is c_1, which is not what we want. So, we multiply everything by $1/z^2$ to shift the coefficients by two places, and get $\operatorname{Res}(z^{-2}f(1/z), 0) = c_{-1} = -\operatorname{Res}(f, \infty)$, as desired. The way to remember the $-1/z^2$ factor is that we are really making a change of variables $w = 1/z$ in the integral we are interested in and the differential $dw = -dz/z^2$ is what gives the $-1/z^2$.

It is often easier to calculate a single residue at ∞ than to calculate several residues at finite points. See the book for more examples and problems.