(1) We have \((-2)^p = e^{p \log(-2)}\) for \(p = -1, \frac{1}{4}, i\).

Writing \(-2 = 2e^{i\pi} = 2e^{i(\pi + 2\pi k)}\) \(k \in \mathbb{Z}\), in polar form, we have
\[
\log(-2) = \log(2) + i(\pi + 2\pi k), \quad k \in \mathbb{Z}.
\]

\(p = -1\)
\[
(-2)^{-1} = e^{-(\log(2) + i(\pi + 2\pi k))} = e^{-\log(2) - i\pi - 2\pi k}
\]
\[
= e^{-\log(2)} e^{-i\pi} e^{-2\pi k}
\]
\[
= \frac{1}{2} \text{. Since } e^{2\pi^2 k} = 1. \quad \text{One value}
\]
\[(\text{You can also just do this by using the definition } (\text{\(-2\)})^{-1} = \frac{1}{\sqrt{2}}
\]
\[(\text{Since } -1 \text{ is an integer})\]

\(p = \frac{1}{4}\)
\[
(-2)^{\frac{1}{4}} = e^{\frac{1}{4} \log(2) + i\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)}
\]
\[
= 2^{\frac{1}{4}} e^{i\frac{\pi}{4} + \frac{\pi k}{2}} \quad k \in \mathbb{Z}
\]
\[
= 2^{\frac{1}{4}} e^{i\pi/4}, 2^{\frac{1}{4}} e^{i\pi/2}, 2^{\frac{1}{4}} e^{i3\pi/4}, 2^{\frac{1}{4}} e^{i5\pi/4}
\]
\[
\text{only 4 values since } \frac{\pi k}{2} = 2\pi \text{ for } k = 4.
\]
\[(\text{You can also do this without logs, since } p = \frac{1}{4} \text{ is rational}
\]
\[(\text{and in particular amounts to calculating } 4^{\text{th}} \text{ roots})\]
\[ p = \frac{i}{(z)^i} = e^{i \log 2 + i(\pi + 2\pi k)} \]
\[ = e^{i \log 2} \cdot e^{\frac{\pi^2}{2} (\pi + 2\pi k)} \quad k \in \mathbb{Z} \]
\[ = e^{-\frac{\pi^2}{2} - 2\pi k} e^{i \log 2} \quad k \in \mathbb{Z} \]

In multiple distinct values

Since \( |e^{-\frac{\pi^2}{2} - 2\pi k} e^{i \log 2}| = e^{-\frac{\pi^2}{2}} \)
are distinct for distinct \( k \).

\( \textcircled{2} \)

\[ f(z) = \cos(z) = \frac{e^{iz} + e^{-iz}}{2} = 0 \]

Letting \( z = x + iy \) we have

\[ e^{i(x+iy)} + e^{-i(x+iy)} \]

\[ = e^{ix} e^{-y} + e^{-ix} e^y \]

\[ = e^{ix} e^{-y} + e^{-ix} e^y = 0 \]

So \( e^{ix} e^{-y} = -e^{-ix} e^y = e^{-i(x+i\pi)y} \)

Equating arguments and magnitudes:

\[ x = -x + \pi + 2\pi k \Rightarrow 2x = \pi + 2\pi k \]
\[ \Rightarrow x = \frac{\pi}{2} + \pi k \quad k \in \mathbb{Z} \]

\[ -y = y \Rightarrow \boxed{y = 0} \]

So the zeros are \( \frac{\pi}{2} + \pi k \quad k \in \mathbb{Z} \)
(3)

(a) Parameterize: \( r(t) = 2 + e^{it}, \quad t \in [0, 2\pi] \),

\[
\oint_{|z|=1} \frac{z}{z-1} \, dz = \int_0^{2\pi} (2 + e^{it}) \cdot i e^{it} \, dt
\]

\[
= \int_0^{2\pi} (2 + e^{-it}) \cdot i e^{it} \, dt = i \int_0^{2\pi} 2 e^{it} \, dt + i \int_0^{2\pi} t \, dt
\]

\[
= \frac{i 2 e^{it}}{i} \bigg|_0^{2\pi} + i t \bigg|_0^{2\pi} = 2\pi i.
\]

(b) Letting \( f(z) = e^{z^2} \), analytic everywhere, Cauchy’s Integral Formula tells us that

\[
\oint_{|z|=1} \frac{f(z)}{(z-1)^2} \, dz = \frac{1}{4} \int \frac{f(z)}{(z-\frac{1}{2})^2} \, dz
\]

\[
= \frac{1}{4} \cdot 2\pi i \cdot f'(\frac{1}{2}) = 2\pi i \cdot \frac{e^{\frac{1}{4}} \cdot 2 \frac{1}{2}}{2}\bigg|_{z=\frac{1}{2}}
\]

\[
= \frac{\pi i}{2} \cdot e^{\frac{1}{4}}.
\]
\( f(z) = \frac{e^{\pi z}}{(z-\pi)^2} \) has:

- a pole of order 2 at \( z=\pi \)

Since \( f(z) = \frac{g(z)}{(z-\pi)^2} \) for \( g(z) = e^{\pi z} \)

and \( g(z) \) is analytic at \( z=\pi \)

(since \( \frac{\pi}{z} \) is analytic at \( \pi \)

\( e^{\pi z} \) is too)

and \( g(\pi) = e^{\pi \pi} \neq 0 \).

The residue may be calculated as:

\[
\text{Res}(\pi) = \lim_{z \to \pi} \frac{d}{dz} \left( (z-\pi)^2 \frac{e^{\pi z}}{(z-\pi)^2} \right)
\]

\[
= \lim_{z \to \pi} e^{\pi z} \cdot \left( -\frac{\pi}{e^2} \right)
\]

\[
= e \left( -\frac{\pi}{e^2} \right) = \frac{-e}{\pi}
\]

- An essential singularity at \( z=0 \), since

\[
\frac{e^{\pi z}}{(z-\pi)^2} = \frac{1}{\pi^2 (1-\frac{z}{\pi})^2} \cdot e^{\pi z}
\]

\[
= \frac{1}{\pi^2} \left( 1 + \frac{z}{\pi} + \frac{z^2}{\pi^2} + \cdots \right) \left( 1 + \frac{\pi}{2} \cdot \frac{z^2}{\pi^2} + \frac{1}{2!} \frac{z^3}{\pi^3} + \cdots \right)
\]

has infinitely many \( \frac{1}{z^n} \) terms.
\[ I = \int_{-\infty}^{\infty} \frac{\cos(2x)}{(1 + x^2)^2} \, dx = \text{Re} \int_{-\infty}^{\infty} \frac{e^{i2z}}{(1 + z^2)^2} \, dz. \]

\[ = \text{Re} \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{i2z}}{(1 + z^2)^2} \, dz. \]

Since \(1 + z^2\) has no real zeros, we close this with a simple semicircle contour:

\[ \gamma_R(t) = R e^{it}, \quad t \in [0, \pi] \]
\[ C_R = \gamma_R + \gamma_1 \]
\[ f(z) = \frac{e^{i2z}}{(z+i)^2(z-i)^2} \] has poles of order 2 at \(z = \pm i\).

For this contour we have \( \int f(z) \, dz \)
\[ C_R = 2\pi i \text{ Res } (i) \]

and \( \text{ Res } (i) = \lim_{z \to i} \frac{d}{dz} \frac{(z-i)^2 e^{i2z}}{(z+i)^2(z-i)^2} \]
\[ = \lim_{z \to i} \frac{2i e^{i2z}}{(z+i)^2} + \frac{e^{i2z}}{(z+i)^3} \]
\[ = \frac{2i e^{-2}}{(-1)^2} - \frac{2i e^{-2}}{(-1)^3} = \frac{2i e^{-2}}{4} - \frac{2e^{-2}}{8i} \]
\[ = 2e^{-2} \left( \frac{-i}{4} - \frac{i}{8} \right) = 2e^{-2} \left( -\frac{3}{8}i \right). \]

So \[ \int_{C_R} f(z) \, dz = 2\pi i \cdot 2e^{-2} \left( -\frac{3}{8}i \right) \]

\[ = \frac{3\pi}{2e^2}. \]

We have
\[
\lim_{R \to \infty} \int_{C_R} f(z) \, dz = \lim_{R \to \infty} \int_{-R}^{R} f(z) \, dz
\]

The latter integral is
\[
\int_{C_R} \frac{e^{iz}}{(1+z^2)^2} \, dz \leq \max_{z \in C_R} \left| \frac{e^{iz}}{(1+z^2)^2} \right| \cdot \pi R
\]

for \( z = x + iy \) on \( C_R \), we have
\[
\left| e^{(x+iy)} \right| = \left| e^{ix} \cdot e^{-2iy} \right|
\]

\[ \leq 1 \quad \text{since} \quad y \geq 0. \]

and
\[ \left| \frac{1}{(1+z^2)^2} \right| = \frac{1}{|z^4 + 2z^2 + 1|} \]

\[ \leq \frac{1}{|z|^4 - 2|z|^2 - 1} = \frac{1}{R^4 - 2R^2 - 1} = O \left( \frac{1}{R^4} \right). \]
Thus, \( \int_{\gamma_R} f(z) dz = O \left( \frac{1}{R^4} \right) \). \( \pi R = O \left( \frac{1}{R^3} \right) \to 0 \) as \( R \to \infty \).

and we have \( I = \frac{3\pi}{2e^2} \).

On the exam, it is fine to justify
\[
\int_{\gamma_R} \frac{p(z)}{q(z)} e^{iaz} dz \to 0 \text{ as } R \to \infty
\]
for polynomials \( p, q \)
simply by citing Jordan's lemma.

But be sure to check that \( \deg(q) > \deg(p) + 1 \),
and to be careful about the sign of \( a \).