1 The Triangle Inequality Argument

When applying the residue theorem to compute definite integrals, we are often faced with a situation where we have to show that the integral of some function on a semicircular contour \( \gamma(t) = Re^{it}, t \in [0, \pi] \) in the upper half plane vanishes as \( R \to \infty \).

The simplest way to do this, which works for instance when \( f(z) = P(z)e^{iaz}/Q(z) \) for polynomials \( P, Q \) satisfying \( \deg(Q) \geq \deg(P)+2 \), is to use the triangle inequality for integrals. This gives the bound:

\[
\left| \int_{\gamma_R} \frac{P(z)e^{iaz}}{Q(z)} \, dz \right| \leq \int_{0}^{\pi} \left| \frac{P(\gamma(t))e^{i\gamma(t)}}{Q(\gamma(t))} \right| |\gamma'(t)| \, dt \leq \max_{z \in \gamma_R} \left( \frac{|P(z)|}{|Q(z)|} \cdot |e^{iaz}| \right) \cdot \pi R.
\]

When \( a > 0 \), we observe that every \( z = x + iy \) on \( \gamma_R \) has \( y > 0 \), which gives the absolute value bound:

\[
|e^{iaz}| = |e^{iay}e^{ay}| = 1 \cdot e^{-ay} \leq 1.
\]

Note that this only works for \( a > 0 \); for \( a < 0 \) we would have to look at a contour in the lower half plane (i.e., \( y < 0 \)).

Since \( Q \) has degree at least two larger than \( Q \), applications of the triangle inequality for addition of complex numbers:

\[
|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|,
\]

can be used to isolate the higher order terms, and conclude that

\[
\left| \frac{P(z)}{Q(z)} \right| \leq O(1/R^2),
\]

for large \( R \). Plugging these estimates into (\( \ast \)) tells us that the value of the integral is \( O(1/R) \), which vanishes as \( R \to \infty \).
2 Jordan’s Lemma

It turns out that the above argument is wasteful; we only used that $|e^{-ay}| \leq 1$ for positive $y$, but most of the time, it is actually much smaller! By doing a more delicate calculation, we can make the above argument work in the more general case when $\deg(Q) \geq \deg(P) + 1$, rather than 2, and this is quite useful.

The more delicate bound is called Jordan’s lemma, and it says that

$$\int_0^\pi e^{-R \sin \theta} d\theta \leq \frac{\pi}{R}.$$  

Before proving this, let’s see why it is useful. Suppose I have some function $f(z)$, such as $f(z) = P(z)/Q(z)$ for polynomials $\deg(Q) \geq \deg(P) + 1$, whose maximum value on $\gamma_R$ is $M_R$ and $M_R \to 0$ as $R \to \infty$. Then, the triangle inequality tells us that

$$\left| \int_{\gamma_R} f(z) e^{iaz} dz \right| \leq \int_0^\pi |f(Re^{it})||e^{iR(\cos t + i \sin t)}||iRe^{it}|dt \leq M_R \cdot R \int_0^\pi e^{-aR \sin t} dt \leq M_R \cdot R \cdot \frac{\pi}{aR} \to 0,$$

as $R \to \infty$.

Now for the proof. The key observation is that on the interval $[0, \pi/2]$, the function $\sin \theta$ is at least $\frac{\theta}{\pi/2}$ — this is easy to see graphically, by plotting $y = \sin x$ and noticing that the line $y = \frac{x}{\pi/2}$ connecting the points $(0,0)$ and $(\pi/2,1)$ lies strictly below the curve\(^1\). Thus, $e^{-R \sin \theta} \leq e^{-2R\theta/\pi}$ for $\theta \in [0, \pi/2]$, and we have

$$\int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{e^{-2R(\pi/2)/\pi}}{-2R/\pi} - \frac{e^0}{-2R/\pi} = \frac{\pi}{2R} (1 - e^{-R}) \leq \frac{\pi}{2R}.$$  

Since $\sin(\theta)$ is symmetric about $\pi/2$, we can repeat the same argument on the interval $[\pi/2, \pi]$ to get

$$\int_{\pi/2}^\pi e^{-R \sin \theta} d\theta \leq \frac{\pi}{2R},$$

and adding the two gives Jordan’s lemma.

All of the above arguments can be adapted to the case $a < 0$, but we must use a contour in the lower half plane ($y < 0$) rather than in the upper half plane.

\(^1\)It can also be proven analytically by noticing that $\sin \theta = \frac{a}{\pi/2}$ at the endpoints $\theta = 0, \pi/2$, and that $(\sin \theta)'' = -\sin \theta \leq 0$ in the interval $[0, \pi/2]$ so the function is concave in this interval.
3 Cauchy Principal Values

A Cauchy Principal value is obtained by a specific way of defining an improper integral as a limit of proper integrals. Traditionally, we learn that an improper integral of the first kind is the following limit of definite integrals:

$$\int_{0}^{\infty} f(x)dx = \lim_{b \to \infty} \int_{0}^{b} f(x)dx,$$

so that we may define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a \to -\infty} \int_{a}^{0} f(x)dx + \lim_{b \to \infty} \int_{0}^{b} f(x)dx.$$

Sometimes, neither of the integrals on the right hand side exist (for instance, take \(f(x) = \sin(x)\)), but there is a sense in which the integral on the left hand side does. For instance, it makes sense to assert that

$$\int_{-\infty}^{\infty} \sin(x) = 0$$

since \(\sin(-x) = -\sin(x)\), i.e., the function is odd, and the contributions from positive and negative \(x\) cancel. This can be made formal by taking the limit symmetrically, and this is what we mean by the principal value:

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)dx := \lim_{R \to \infty} \int_{-R}^{R} f(x)dx,$$

which does indeed give zero in the case of \(f(x) = \sin(x)\). Such integrals are often useful, and are moreover easily amenable to contour integration techniques because we can close the contour with a semicircle whose center is fixed at zero, rather than having to consider semicircles whose centers vary as their radii increase.

Similarly, for improper integrals of the second kind, such as

$$\int_{-1}^{1} \frac{1}{x} dx,$$

the principal value is defined by taking the limit symmetrically about the point of discontinuity:

$$\text{p.v.} \int_{-1}^{1} \frac{1}{x} dx := \lim_{\epsilon \to 0^+} \left( \int_{-1}^{0-\epsilon} \frac{1}{x} dx + \int_{0+\epsilon}^{1} \frac{1}{x} dx \right),$$

rather than separately on each side. When closing the contour, this corresponds to semicircles of shrinking radius \(\epsilon > 0\) centered at the point of discontinuity.