# Math 121A Spring 2015, Homework 9

*Due April 14 at 10am in my office, or April 13 in class*

We will use the notation $\hat{f}(n)$ to denote the $n$th (exponential) Fourier coefficient of $f$, i.e.,

$$f(x) = \sum_n \hat{f}(n) e^{inx}.$$

1. Determine which of the following functions on the real line are periodic. If they are, determine the fundamental period; if not, explain why.

$$\sin(\pi x), e^x, e^{ix} + \sin(x), \sin(3x) + \cos(5x), \sin(3x) + \sin(\sqrt{5}x).$$

2. If $f : \mathbb{R} \to \mathbb{C}$ is periodic with period $2L$ (i.e., $f(x) = f(x + 2L)$) what are the periods of $g(x) = f(cx)$, $h(x) = f(x - t)$, and $k(x) = f(-x)$?

   Calculate $\hat{h}(n)$ and $\hat{k}(n)$ in terms of $\hat{f}(n)$.

3. Calculate the integrals:

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx, \quad \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx$$

   for all integer values of $m$ and $n$. Then calculate

$$\int_{-1}^{1} \sin nx \sin mx dx, \quad \int_{-1}^{1} e^{inx} e^{-imx} dx.$$

4. Boas 4.1, 5.10, 6.10, 7.10, 7.11, 8.11, 8.12, 8.13.

5. (a) Evaluate the (exponential) Fourier coefficients $\hat{f}(n)$ of the sawtooth function:

$$f(x) = x, \quad -\pi \leq x < \pi.$$

   (b) Use Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

   (c) Use a similar method applied to the function

$$f(x) = x^2 \quad -\pi \leq x < \pi$$

   to find the sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

6. Show that if $f$ has only finitely many nonzero Fourier coefficients $\hat{f}(n) \neq 0$, then $f$ must be infinitely differentiable. Conclude that the sawtooth function must have infinitely many nonzero Fourier coefficients.

7. (Optional) Suppose $f : [0, 2\pi] \to \mathbb{C}$ is given by $f(\theta) = F(e^{i\theta})$ where $F(z)$ is analytic on the unit circle $|z| = 1$. Show that the Laurent series for $F$ in an annulus containing the unit circle may be used to compute the (exponential) Fourier series of $f$. What happens if $F$ is analytic on and inside the unit circle?

8. (Optional) Suppose that the partial sums of the Fourier series of $f \in L^2[-\pi, \pi]$ are

$$S_N = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$

Show that for any coefficients $d_n$,

$$\|f - S_N\|^2 \leq \left\| f - \sum_{n=-N}^{N} d_n e^{inx} \right\|^2,$$

i.e., the partial sums of the Fourier series minimize the mean square error among all linear combinations of $e^{-iNx}, \ldots, e^{iNx}$.

*(hint: expand*

$$\left\| f - \sum_{n=-N}^{N} d_n e^{inx} \right\|^2 = \left\langle f - \sum_{n=-N}^{N} d_n e^{inx} | f - \sum_{n=-N}^{N} d_n e^{inx} \right\rangle,$$

*expand $f$ as a Fourier series, and use orthogonality of the exponential functions.)*