Homework 8 solutions

(1) for $z = re^{i\theta}$, $\theta \in [0, 2\pi)$, $r \geq 0$, we have

$$\operatorname{Log} z = \operatorname{Log} r + i\theta \qquad \text{So } u = \operatorname{Log} r$$
$$v = \theta, \quad \theta \in [0, 2\pi)$$

When $\theta \neq 0$ we have:

$$u_r = \frac{1}{r} \qquad v_r = 0$$

$$u_\theta = 0 \qquad v_\theta = 1$$

whence: $u_r = \frac{1}{r} v_\theta$

and $u_\theta = -\frac{1}{r} v_r$ which are the Polar form of the Cauchy-Riemann equations.

When $\theta = 0$, the function $V(r, \theta) = \theta \pmod{2\pi}$ is not continuous (and therefore not differentiable), so the C-R equations don't even make sense and $\operatorname{Log}(z)$ is not analytic there.

When $r = 0$ $\operatorname{Log} r$ is undefined.

Section 6

⑤ $$\frac{e^z}{z^2-1} = \frac{e\cdot e^{z-1}}{(z-1)(z+1)}$$

$$= \frac{e}{z-1}\ \frac{e^{z-1}}{2+(z-1)} = \frac{e}{2(z-1)}\ \frac{e^{z-1}}{1+\frac{z-1}{2}}$$

$$= \frac{e}{2}\frac{1}{z-1}\Bigg(\underbrace{1+(z-1)+\frac{(z-1)^2}{2!}\cdots}_{\text{converges everywhere}}\Bigg)\Bigg(\underbrace{1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2\cdots}_{\text{converges } |z-1|<2}\Bigg)$$

$$= \frac{e}{2}\frac{1}{z-1}\left(1+\frac{z-1}{2}+(z-1)^2\left(\frac{1}{2!}+\frac{1}{4}-\frac{1}{2}\right)\cdots\right)$$

$$= \frac{e}{2(z-1)} + \frac{e}{4} + \frac{e}{8}(z-1)\cdots$$

So $\text{Res}(1) = \frac{e}{2}$. //

⑥ $$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5}\cdots$$

for $|z|>0$

So $\underline{\underline{\text{Res}(0) = 1}}$

26 + 26′ Letting $\omega = e^{2\pi i/3}$ we have

$$f(z) = \frac{e^{2\pi i z}}{-(z-1)(z-\omega)(z-\omega^2)}$$

since the roots of $1-z^3$ are the cube roots of 1, which are $1, \omega, \omega^2$

$$= \frac{-\,e^{2\pi i z}/(z-1)(z-\omega^2)}{(z-\omega)}$$

So $z=\omega$ is a simple pole, since the denominator is nonzero and analytic at $z=\omega$.

Thus: $$\operatorname{Res}(\omega) = \lim_{z\to\omega} \cancel{(z-\omega)} \frac{-\frac{e^{2\pi i z}}{(z-1)(z-\omega^2)}}{\cancel{(z-\omega)}}$$

$$= \frac{-e^{2\pi i \omega}}{(\omega-1)(\omega-\omega^2)}$$

$$= \frac{-\,e^{2\pi i\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)}}{\omega^2-\omega^3-\omega+\omega^2}$$

$$= \frac{-\,e^{2\pi i(-1/2)}\,e^{-2\pi\frac{\sqrt{3}}{2}}}{3\omega^2-(\omega+\omega^2+\omega^3)}$$

$$= \frac{-\,e^{-i\pi}\,e^{-\pi\sqrt{3}}}{3e^{4\pi i/3}}$$ since $\omega+\omega^2+\omega^3=0$ by HW6

$$= \frac{e^{-\pi\sqrt{3}}}{3}\,e^{-4\pi i/3}$$

$$= \frac{e^{-\pi\sqrt{3}}}{3}\,e^{2\pi i/3} \quad //$$

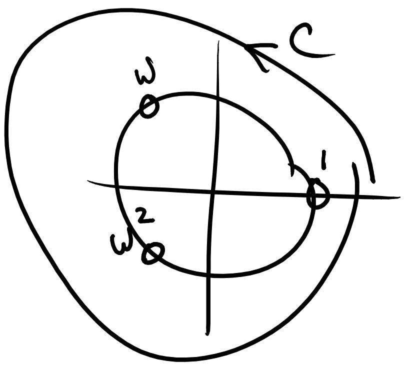
The circle of radius $3/2$ encloses all three singularities,

so we have

$$\frac{1}{2\pi i}\oint_{|z|=3/2} f(z)\,dz = \mathrm{Res}(1) + \mathrm{Res}(\omega) + \mathrm{Res}(\omega^2).$$

$$\mathrm{Res}(\omega) = \frac{e^{-\pi\sqrt{3}}}{3}\, e^{i2\pi/3} \quad \text{(previous page)}$$

$$\mathrm{Res}(1) = \lim_{z\to 1} \frac{-e^{2\pi i z}}{(z-\omega)(z-\omega^2)}$$



$$= \frac{-1}{(1-\omega)(1-\omega^2)} = \frac{-1}{1-\omega^2-\omega+\omega^3}$$

$$= \frac{-1}{3-(1+\omega+\omega^2)} = -\frac{1}{3}$$

$$\mathrm{Res}(\omega^2) = \lim_{z\to\omega^2} \frac{-e^{2\pi i z}}{(z-1)(z-\omega)}$$

$$= \frac{-e^{2\pi i \omega^2}}{(\omega^2-1)(\omega^2-\omega)}$$

$$= \frac{e^{\pi\sqrt{3}}}{\omega^4-\omega^3-\omega^2+\omega}$$

$$= \frac{e^{\pi\sqrt{3}}}{3\omega-(\omega+\omega^2+1)}$$

$$= \frac{e^{\pi\sqrt{3}}}{3\omega}$$

$$e^{2\pi i\omega} = e^{2\pi i(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})} = e^{-i\pi}e^{-2\pi\frac{\sqrt{3}}{2}} = -e^{-\pi\sqrt{3}},$$

$$e^{2\pi i\omega^2} = e^{2\pi i(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3})} = e^{-i\pi}e^{-2\pi(-\frac{\sqrt{3}}{2})} = -e^{\pi\sqrt{3}}$$

So $\oint f(z)\,dz = 2\pi i \left(-\frac{1}{3} + \frac{e^{-\pi\sqrt{3}}}{3} e^{i2\pi/3} + \frac{e^{\pi\sqrt{3}}}{3} e^{-i2\pi/3} \right)$

$$= \frac{2\pi i}{3}\Big(-1 + e^{-\pi\sqrt{3}} \cos\frac{2\pi}{3} + i e^{-\pi\sqrt{3}} \sin 2\pi/3$$
$$+ e^{\pi\sqrt{3}} \cos(2\pi/3)$$
$$- i e^{\pi\sqrt{3}} \sin(2\pi/3)\Big)$$

$$= \frac{2\pi i}{3}\left(-1 - \cosh(\pi\sqrt{3}) - i\sqrt{3}\sinh(\pi\sqrt{3}) \right)$$

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$f(z) = \dfrac{e^{iz}}{(z+2i)^2 (z-2i)^2}$ has a pole of order 2 at $z = 2i$.

$$\text{Res}(2i) = \lim_{z\to 2i} \frac{d}{dz} \cancel{(z-2i)^2} \frac{e^{iz}}{(z+2i)^2 \cancel{(z-2i)^2}}$$

$$= \lim_{z\to 2i} \frac{ie^{iz}}{(z+2i)^2} + \frac{e^{iz}(-2)}{(z+2i)^3}$$

$$= \frac{ie^{-2}}{(4i)^2} - \frac{2e^{-2}}{(4i)^3} = \frac{-4e^{-2} - 2e^{-2}}{(4i)^3}$$

$$= \frac{6e^{-2}}{64i} = \underline{\underline{\frac{3e^{-2}}{32i}}}$$

There are no singularities inside $|z| = 3/2$ so

$$\oint_{|z|=3/2} f(z)\,dz = \underline{\underline{0}}.$$

<u>Problem 3</u>

The singularities occur at the zeros of $\sin(z)$, which are at πn, $n \in \mathbb{Z}$.

These are simple poles, so the residue at πn is.

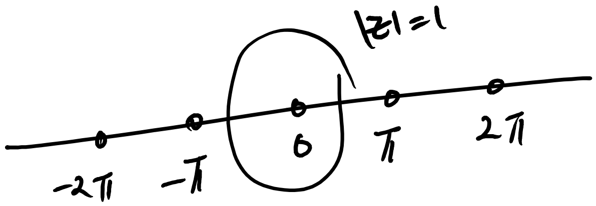
$$\mathrm{Res}(\pi n) = \lim_{z \to \pi n} \frac{(z - \pi n)}{\sin(z)}$$

$$= \lim_{z \to \pi n} \frac{(z - \pi n)}{\sin(z - \pi n + \pi n)}$$

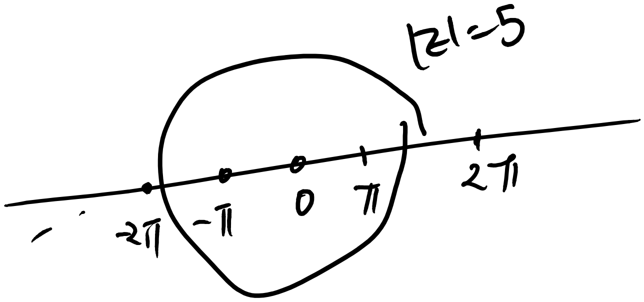
$$= \lim_{z \to \pi n} \frac{(z - \pi n)}{\sin(z - \pi n)(-1)^n}$$

since $\sin(w + n\pi) = \begin{cases} \sin(w) & \text{for even } n \\ -\sin w & \text{for odd } n \end{cases}$

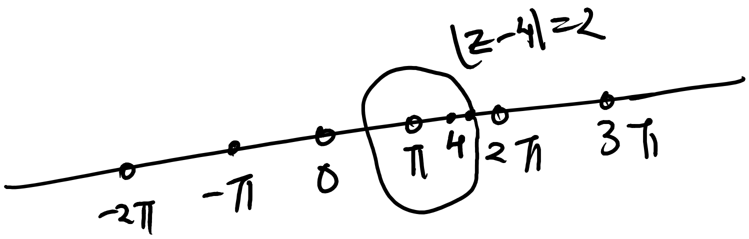
$$= \underline{\underline{(-1)^n}}.$$



$$\oint_{|z|=1} \frac{dz}{\sin(z)} = 2\pi i\,(1) = 2\pi i$$



$$\oint_{|z|=5} = 2\pi i\,(1-2) = \underline{\underline{-2\pi i}}$$



$$\oint_{|z-4|=2} = 2\pi i\,(-1) = \underline{\underline{-2\pi i}}$$

Section 7

(2) Substitute $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta$

$$\cos\theta = \frac{z+z^{-1}}{2}$$

$$\text{So } I = \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \oint_{|z|=1} \frac{1}{iz} \frac{dz}{5-\frac{3}{2}\left(z+\frac{1}{z}\right)}$$

$$= -i \oint \frac{1}{\cancel{z}} \frac{dz}{\frac{10z-3z^2-3}{2\cancel{z}}} = 2i \oint \frac{dz}{-(10z-3z^2-3)}$$

$$= 2i \oint \frac{dz}{(3z-1)(z-3)} \qquad \text{Poles at } z = 1/3 \text{ and } z=3.$$

$$\mathrm{Res}(1/3) = \lim_{z\to 1/3} (z-1/3)\frac{1}{(3z-1)(z-3)} = \frac{1}{3\left(\frac{1}{3}-3\right)}$$

$$= -\frac{1}{8} \,//$$

$$\text{So } I = \cancel{2}\pi i \cdot \cancel{2}i \cdot \left(-\frac{1}{\cancel{8}_{2}}\right) = \frac{\pi}{2} \,//$$

⑦ As in the previous problem, we take $z = e^{i\theta}$

so $\cos\theta = \frac{1}{2}(z + z^{-1})$ and $\cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$. Then

$$I = \oint_{|z|=1} \frac{1}{iz} \frac{\frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)}{5 + \frac{4}{2}\left(z + \frac{1}{z}\right)} \, dz$$

$$= -i \oint \frac{\frac{z^4+1}{z^2}}{10z + 4z^2 + 4} = \frac{-i}{4} \oint \frac{z^4+1}{z^2(z+2)(z+\frac{1}{2})} \, dz$$

Singularities at $z = 0, -2, -\frac{1}{2}$.

$$\mathrm{Res}(0) = \lim_{z\to 0} \frac{d}{dz} \cancel{z^2} \cdot \frac{(z^4+1)}{\cancel{z^2}(z+2)(z+\frac{1}{2})}$$

$$= \lim_{z\to 0} \frac{(4z^3)}{(z+2)(z+\frac{1}{2})} + \frac{(z^4+1)}{(z+2)} \frac{(-1)}{(z+\frac{1}{2})^2}$$

$$+ \frac{(z^4+1)}{(z+\frac{1}{2})} \frac{(-1)}{(z+2)^2}$$

$$= \frac{-1}{2 \cdot \frac{1}{4}} - \frac{1}{\frac{1}{2} \cdot 4} = -2 - \frac{1}{2} = \underline{\underline{-\frac{5}{2}}}$$

$$\mathrm{Res}(-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} \cancel{(z+\tfrac{1}{2})} \frac{(z^4+1)}{(z+2)\cancel{(z+\frac{1}{2})} z^2} = \frac{\frac{1}{16}+1}{\frac{3}{2} \cdot \frac{1}{4}}$$

$$= \frac{17}{16} \cdot \frac{8}{3} = \underline{\underline{\frac{17}{6}}}$$

So $I = 2\pi i \left(\frac{-i}{4}\right)\left(-\frac{5}{2} + \frac{17}{6}\right) = \frac{2}{6} \cdot \frac{\pi}{2}$

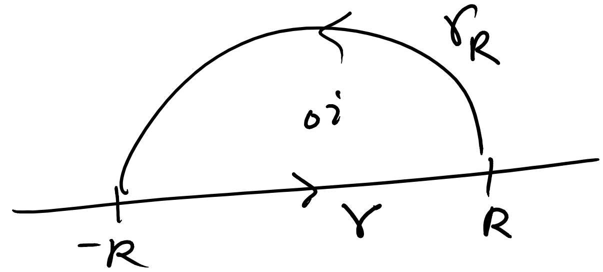
$= \pi/6$ //

(11) $$2I = \int_{-\infty}^{\infty} \frac{dx}{(4x^2+1)^3}$$ since the function is even.

Let $$f(z) = \frac{1}{(4z^2+1)^3} = \frac{1}{64\,(z+\frac{i}{2})^3(z-\frac{i}{2})^3}.$$

Singularities at $z = \pm\frac{i}{2}$

Consider $C_R = \gamma + \gamma_R$



By the Residue theorem:

$$\oint_{C_R} f(z)\,dz = 2\pi i \operatorname{Res}\left(\tfrac{i}{2}\right)$$

$$\operatorname{Res}(i) = \frac{1}{2!}\lim_{z\to i/2} \frac{d^2}{dz^2} \frac{1}{64} \cancel{(z-i/2)^3} \frac{1}{(z+i/2)^3 \cancel{(z-i/2)^3}}$$

$$= \frac{1}{2!}\frac{1}{64} \frac{(-3)(-4)}{(i/2+i/2)^5} = \frac{12}{128}\frac{1}{i} = -\frac{3i}{32}$$

So $$\oint_{C_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\gamma_R} f(z)\,dz = \frac{-6\pi i^2}{32} = \frac{3\pi}{16}$$

where $\gamma_R(t) = Re^{it}$ $t\in[0,\pi)$

By the triangle inequality, $$\int_{\gamma_R} f(z)\,dz \leq \left(\max_{z\in\gamma_R}|f(z)|\right)\pi R$$

Since $|z|=R$ on γ_R, we have

$$|f(z)| = \frac{1}{|4z^2+1|^3} \le \frac{1}{(4|z|^2-1)^3} = \frac{1}{(4R^2-1)^3}$$
$$= O\left(\frac{1}{R^6}\right)$$

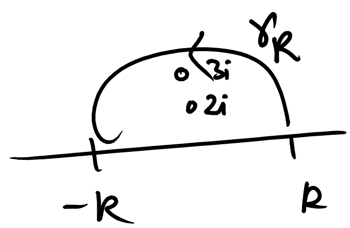
So
$$\int_{\gamma_R} f(z)\,dz = O\left(\frac{1}{R^5}\right) \longrightarrow 0 \quad \text{as } R\to\infty$$

and
$$2I = \lim_{R\to\infty} \oint_{C_R} f(z)\,dz = \frac{3\pi}{16} \Rightarrow \underline{\underline{I = \frac{3\pi}{32}}}$$

(13)
$$2I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx \qquad \text{because even.}$$

Let
$$f(z) = \frac{z^2}{(z^2+4)(z^2+9)} = \frac{z^2}{(z+2i)(z-2i)(z+3i)(z-3i)}.$$

There are no singularities on $\mathbb{R}$ so we use the same semicircle as in the last question:



$$\oint_{C_R} f(z)\,dz = 2\pi i\left(\mathrm{Res}(2i) + \mathrm{Res}(3i)\right)$$

These are simple poles so

$$\mathrm{Res}(2i) = \lim_{z\to 2i} \frac{z^2}{(z+2i)(z-3i)(z+3i)}$$

$$= \frac{4i^2}{(4i)(-i)(5i)} = \frac{-4}{20\,i} = \frac{i}{5}$$

$$\mathrm{Res}(3i) = \lim_{z\to 3i} \frac{z^2}{(z+2i)(z-2i)(z+3i)}$$

$$= \frac{9i^2}{(5i)(i)(6i)} = \frac{-9}{-30i} = -\frac{3}{10}i$$

So $$\oint_{C_R} = 2\pi i \cdot i\left(\frac{1}{5} - \frac{3}{10}\right) = \frac{2\pi}{10} = \frac{\pi}{5}$$

As before, we have $$\max_{z\in\gamma_R} |f(z)| \le \frac{|z|^2}{(|z|^2-4)(|z|^2-9)}$$

$$= O\left(\frac{1}{R^2}\right)$$

So by the triangle ineq., $\int_{\gamma_R} f(z)\,dz \to 0$ as $R\to\infty$.

Thus $I = \pi/10$.

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$$I = \int_{-\infty}^{\infty} \frac{x\sin x \, dx}{x^2+4x+5}. \text{ Take } f(z) = \frac{z e^{iz}}{(z+2-i)(z+2+i)}$$

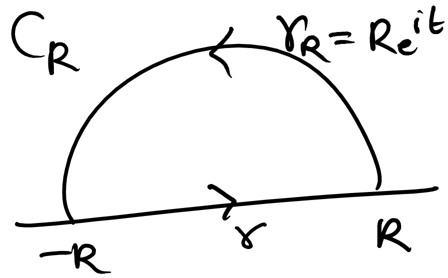
$$x = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

with simple poles at $-2 \pm i$.

Then $I = \text{Im}\left(\int_{-\infty}^{\infty} \frac{z e^{iz}}{(z+2-i)(z+2+i)} dz \right)$.

There are no zeros on $\mathbb{R}$ and the integrand is of type $\frac{p(z)}{q(z)} e^{iaz}$, $a>0$ so we use a semicircle

Contour in the upper halfplane.



There is one pole in the upper halfplane:

$$\text{Res}(-2+i) = \lim_{z \to -2+i} \frac{\cancel{(z+2-i)}\, z e^{iz}}{\cancel{(z+2-i)}(z+2+i)}$$

$$= \frac{(-2+i)\, e^{-2i-1}}{(-\cancel{2}+i+\cancel{2}+i)} =$$

$$= \frac{(-2+i)(\cos(-2) + i\sin(-2))}{e \cdot 2i}$$

$$= \frac{1}{e\cdot 2i}\left(-2\cos(2)+\sin(2)+i\left(\cos(2)+2\sin(2)\right)\right)$$

So $$\mathrm{Im}\left(\oint_{C_R} f(z)\,dz\right) = \mathrm{Im}\left(\frac{\pi}{e}\left(-2\cos(2)+\sin(2) + i\left(\cos(2)+2\sin(2)\right)\right)\right)$$

$$= \frac{\pi}{e}\left(\cos(2)+2\sin(2)\right) \quad //$$

On the other hand, $$\int_{\gamma_R} \frac{z e^{iz}}{z^2+4z+5}\,dz \longrightarrow 0$$

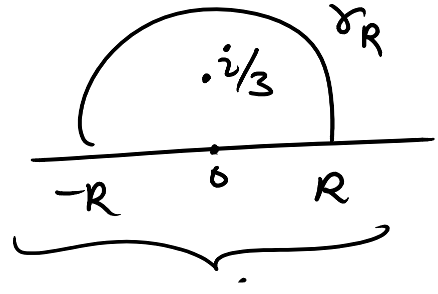
by Jordan's lemma since $\deg(z) \leq \deg(z^2+4z+5) - 1$

So $$I = \frac{\pi}{e}\left(\cos(2)+2\sin(2)\right). \quad //$$

(20) $$2I = \int_{-\infty}^{\infty} \frac{\cos x}{(1+9x^2)^2}\,dx$$ since integrand is even.

Take $$f(z) = \frac{1}{9^2}\frac{e^{iz}}{\left(z+\frac{i}{3}\right)^2\left(z-\frac{i}{3}\right)^2}.$$ This has poles of order 2 at $z = \pm i/3$.

The Problem is now identical to the previous one, except we are interested in the real part (cosine).



Use a semicircle contour and Jordan's lemma to establish that

C_R

$$2I = \int_{-\infty}^{\infty} f(x)\,dx = \mathrm{Re}\left[\lim_{R\to\infty} \oint_{C_R} f(z)\,dz\right] = \mathrm{Re}\left[2\pi i \;\mathrm{Res}(i/3)\right]$$

The residue is:

$$\mathrm{Res}(i/3) = \lim_{z\to i/3} \frac{d}{dz} \cancel{(z-i/3)^2} \frac{1}{81} \frac{e^{iz}}{\cancel{(z-i/3)^2}(z+i/3)^2}$$

$$= \frac{1}{81} \lim_{z\to i/3} \frac{e^{iz}(-2)}{(z+i/3)^3} + \frac{i e^{iz}}{(z+i/3)^2}$$

$$= \frac{1}{81}\left(\frac{-2e^{-1/3}}{8i^3/27} + \frac{i\,e^{-1/3}}{4i^2/9}\right)$$

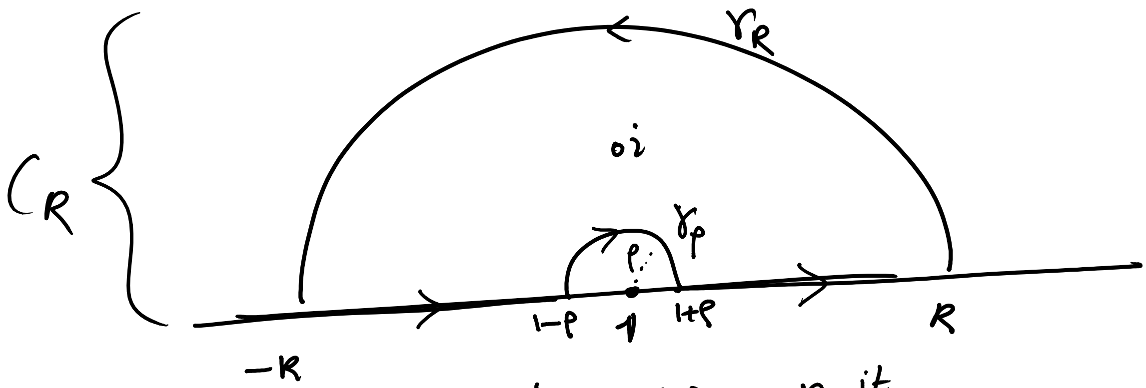
$$= \frac{-i}{9e^{1/3}}. \quad \text{So} \quad 2I = 2\pi i \frac{(-i)}{9e^{1/3}}$$

$$\Rightarrow I = \frac{\pi}{9e^{1/3}}$$

(22) $I = \int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+1)}$. Take $f(z) = \frac{1}{(z-1)(z^2+1)}$,

singularities at $z = 1, \pm i$.

Since $z=1$ is on the real line, we have to use an indented contour to integrate around it:



$\gamma_R(t) = Re^{it}$

$\gamma_\rho(t) = \rho e^{it} + 1, \quad t \in [\pi, 0]$

The only pole in the interior of C_R is at i:

$$\operatorname{Res}(i) = \lim_{z \to i} (z-i) \frac{1}{(z-1)(z^2+1)}$$

$$= \frac{1}{(i-1)(i+i)}, \quad \text{so} \oint_{C_R} f(z)\,dz = 2\pi i \frac{1}{(i-1)(2i)}$$

$$= \frac{\pi}{i-1} \,//$$

On the other hand,

$$\oint_{C_R} f(z)\,dz = \left(\int_{-R}^{1-\rho} + \int_{1+\rho}^{R} + \int_{\gamma_\rho} + \int_{\gamma_R} \right) f(z)\,dz.$$

We have $\left| \int_{\gamma_R} f(z)\,dz \right| \leq O\left(\frac{1}{R^3}\right) \cdot \pi R \longrightarrow 0$ as $R \to \infty$ by a triangle inequality argument.

However, $\lim_{\rho \to 0} \int_{\gamma_\rho} f(z)\,dz = \lim_{\rho \to 0} \int_{\pi}^{0} \frac{1}{(\cancel{\rho e^{it}})(\rho^2 e^{2it} + 2\rho e^{it} + 1 + 1)} \cdot i\cancel{\rho e^{it}}\,dt$

$$= i \int_{\pi}^{0} \lim_{\rho \to 0} \frac{1}{(\rho^2 e^{2it} + 2\rho e^{it} + 2)}\,dt = i \int_{\pi}^{0} \frac{1}{2}\,dt = -\frac{i\pi}{2} \,//$$

(Such semicircular integrals also yield half the residue with the appropriate orientation, so we can also use

$$\frac{1}{2\pi i}\int_{\gamma_\rho} f(z)\,dz = -\frac{1}{2}\,\text{Res}\,(1) = -\frac{1}{2}\lim (z-1)\frac{1}{(z-1)(1+z^2)}$$

$$= -\tfrac{1}{2}\,\frac{1}{2} = -\tfrac{1}{4}\,,\ )$$

which gives the same answer.

Thus, $I = \lim\limits_{\substack{\rho\to 0^+\\ R\to\infty}} \left(\int_{-R}^{1-\rho} + \int_{1+\rho}^{R}\right) f(z)\,dz =$

$$\lim_{\substack{\rho\to 0\\ R\to\infty}} \left(-\int_{\gamma_\rho} f(z)\,dz + \oint_{C_R} f(z)\,dz\right)$$

$$= \frac{i\pi}{2} + \left(\frac{\pi}{-1+i}\right)$$

$$= \pi\left(\frac{i}{2} + \frac{1}{-1+i}\right)$$

$$= \pi\left(\frac{-i-1+2}{-2i+2}\right)$$

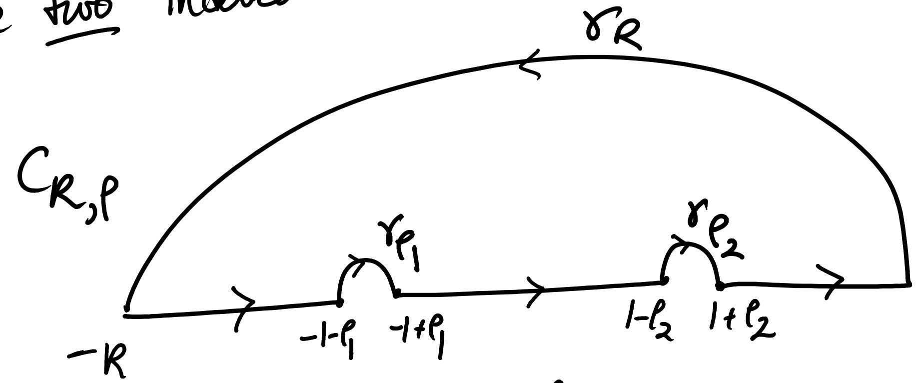
$$= \pi\left(\frac{-i-1}{(-i-1)2}\right) = \underline{\underline{\pi/2}}$$

(24) $$I = \int_{-\infty}^{\infty} \frac{x \sin \pi x}{1-x^2}\,dx = \operatorname{Im}\left[\int_{-\infty}^{\infty} \frac{z e^{i\pi z}}{1-z^2}\,dz\right]$$

The function $f(z) = \dfrac{z e^{i\pi z}}{(1-z)(1+z)}$ has two singularities,

at $z = \pm 1$, both on the real line.

So we have to use two indented contours:



f is analytic on the interior of $C_{R,\rho}$ so $\displaystyle\oint_{C_{R,\rho}} f(z)\,dz = 0$.

On the other hand,

$$0 = \oint f(z)\,dz = \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_R} + \underbrace{\int_{-R}^{-1-\rho_1} + \int_{-1+\rho_1}^{1-\rho_2} + \int_{1+\rho_2}^{R}}_{I_{R,\rho}}\right) f(z)\,dz$$

We are interested in the sum of the last three integrals, as $\rho_1, \rho_2 \to 0$ and $R \to \infty$. Rearranging gives:

$$\lim_{\substack{\rho_1,\rho_2\to 0\\ R\to\infty}} I_{R,\rho} = -\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz - \lim_{\rho_1\to 0}\int_{\gamma_{\rho_1}} f(z)\,dz$$

$$- \lim_{\rho_2 \to 0} \int_{\gamma_{\rho_2}} f(z)\, dz.$$

The limit of the first integral is zero by Jordan's lemma, since $f(z) = \frac{P(z)e^{i\pi z}}{Q(z)}$ for $Q(z) = 1 - z^2$ of degree strictly greater than $P(z) = z$.

For the other two integrals, we proceed as in the previous qestion:

$$\lim_{\rho_1 \to 0} \int_{\gamma_{\rho_1}} \frac{z e^{i\pi z}}{(1-z)(1+z)} \qquad z = -1 + \rho e^{it} \quad t \in [\pi, 0]$$

$$= \lim_{\rho_1 \to 0} \int_{\pi}^{0} \frac{(-1+\rho e^{it})\left(e^{-i\pi + i\pi\rho e^{it}}\right) i\cancel{\rho e^{it}}}{(2-\rho e^{it})\cancel{(\rho e^{it})}}\, dt$$

$$= i \int_{\pi}^{0} \lim_{\rho_1 \to 0} \frac{(-1+\rho e^{it})\left(e^{-i\pi + i\pi\rho e^{it}}\right)}{(2-\rho e^{it})}\, dt$$

$$= i \int_{\pi}^{0} \frac{(-1)(e^{-i\pi})}{2}\, dt = \frac{i}{2}(-\pi).$$

Or, we can use the observation that integrating along a shrinking semicircle gives half the residue, with the sign flipped because γ_{ρ_2} is negatively oriented:

$$\lim_{\rho_2\to 0}\int_{\gamma_{\rho_1}} f(z)\,dz = (-i\pi)\,\mathrm{Res}(f,1)$$

$$= -i\pi \lim_{z\to 1} \frac{\overset{-1}{\cancel{(z-1)}}\, z e^{i\pi z}}{\cancel{(1-z)}(z+1)}$$

$$= -i\pi\, \frac{(-1)(e^{i\pi})}{2} = \frac{-i\pi}{2} \,//$$

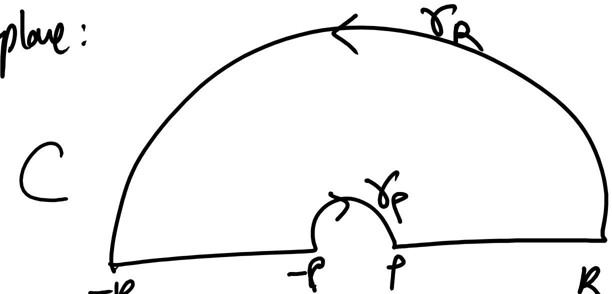
Thus, we must have

$$I = \mathrm{Im}\left[\lim_{\substack{\rho\to 0\\ R\to\infty}} I_{R,\rho}\right] = -\left(-\frac{i\pi}{2}\right) - \left(-\frac{i\pi}{2}\right) = \; = \mathrm{Im}[i\pi] = \underline{\underline{\pi}}.$$

(29) When $a=0$, $\int_0^\infty 0\,dx = 0$. Otherwise, because the function is even:

$$2I = \int_{-\infty}^{\infty} \frac{\sin ax}{x}\,dx = \mathrm{Im}\left[\int_{-\infty}^{\infty} \frac{e^{iaz}}{z}\,dz\right].$$

<u>Case $a>0$</u>: We use an indented contour around $z=0$, and a semicircle in the upper halfplane:



Let $f(z) = \dfrac{e^{iaz}}{z}$.

f is analytic inside C so

$$\oint_C f(z)\,dz = \left(\int_{-R}^{-\rho} + \int_{\rho}^{R} + \int_{\gamma_\rho} + \int_{\gamma_R}\right) f(z)\,dz$$

$$\oint_C \; \underbrace{\left(\int_{-R} \;+\; \int_{\rho}\right)}_{I_{R,\rho}} \;+\; \int_{\gamma_\rho} \;+\; \int_{\gamma_R} f(t)\,dt$$

$\displaystyle\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz$ vanishes by Jordan's lemma since $a>0$.

and $\displaystyle\lim_{\rho\to 0}\int_{\gamma_\rho}\frac{e^{iaz}}{z} = -i\pi\ \text{Res}(0)$ (by an argument similar to the last two problems)

$$= -i\pi\left(\lim_{z\to 0} z\frac{e^{iaz}}{z}\right)$$

$$= -i\pi e^{i0} = \underline{\underline{-i\pi}}.$$

Thus, we have

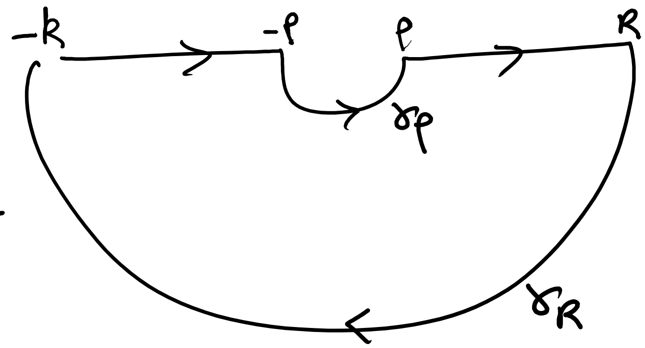
$$2I = \text{Im}\left[\lim_{\substack{R\to\infty\\ \rho\to 0}} I_{R,\rho}\right] = \text{Im}\left[-\lim_{\rho\to 0}\int_{\gamma_\rho} f(z)\,dz\right] = \text{Im}\left[i\pi\right] = \pi$$

so $\underline{\underline{I = \pi/2}}$.

<u>Case $a<0$</u>: Here, we have to take a semicircle and indented contour in the <u>lower</u> halfplane since e^{iaz} for $a<0$ is small when $z = x+iy$, $y\to-\infty$.

Again,

$$\oint_C = \int_{-R}^{-\rho} + \int_{\rho}^{R} + \int_{\gamma_R} + \int_{\gamma_\rho}.$$

C 

Now: $\displaystyle I_{\gamma_R} = \int_{\gamma_R}\frac{e^{iaz}}{z}\,dz = \int_0^{-\pi}\frac{e^{iaR(\cos t + i\sin t)}\,i\cancel{Re^{it}}}{\cancel{Re^{it}}}\,dt$

$$\left(\gamma_R(t) = Re^{it} \quad t \in [0, -\pi] \right)$$

$$= i\int_0^{-\pi} e^{iaR\cos t}\, e^{-a\sin(t)}\,dt$$

Changing t to -t and applying the triangle inequality, we have

$$|I_{\gamma_R}| \leq |i| \int_0^{\pi} |e^{iaR\cos(t)}|\,|e^{-aR\sin(-t)}|\,dt$$

$$= \int_0^{\pi} e^{a\sin(t)}\,dt \leq \frac{\pi}{-aR} \text{ by Jordan's lemma}$$

$$\longrightarrow 0 \text{ as } R \longrightarrow \infty.$$

The small semicircle acquires the opposite sign since we are now integrating in the positive direction:

$$\lim_{\rho\to 0}\int_{\gamma_\rho}\frac{e^{iaz}}{z}\,dz = \lim_{\rho\to 0}\int_{-\pi}^{0}\frac{e^{ia\rho e^{it}}}{\cancel{\rho e^{it}}}\, i\cancel{\rho e^{it}}\,dt$$

$$= i\int_{-\pi}^{0}\lim_{\rho\to 0} e^{ia\rho e^{it}}\,dt$$

$$= i\int_{-\pi}^{0} dt = \underline{\underline{i\pi}}$$

So in this case we have $2I = \mathrm{Im}\left[-\int_{\gamma_\rho} f(z)\,dz\right] = \mathrm{Im}\left[-i\pi\right]$

$$= -\pi$$

$$\Rightarrow \underline{\underline{I = \pi/2}}$$