Some Graded Homework 7 Problems

Chapter 14, Section 3, Problem 22.

Let

$$\text{☻} := \oint_C \frac{\sin(2z)\,dz}{(6z-\pi)^3},$$

where C is the (positively-oriented) circle $\{z \in \mathbb{C} : |z| = 3\}$. Using Problem #21 with $n = 2$, $a = \frac{\pi}{6}$, and $f(z) = \sin(2z)$, we have $f''(\frac{\pi}{6}) = \frac{2!}{2\pi i} \oint_C \frac{\sin(2z)\,dz}{(z-\frac{\pi}{6})^3}$. Since $f''(z) = -4\sin(2z)$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, we must have

$$\oint_C \frac{\sin(2z)\,dz}{(z-\frac{\pi}{6})^3} \;=\; \frac{2\pi i}{2!}(-4\frac{\sqrt{3}}{2}) \;=\; -2\sqrt{3}\pi i.$$

So,

$$\text{☻} \;=\; \frac{1}{6^3}\int_C \frac{\sin(2z)\,dz}{(z-\frac{\pi}{6})^3} \;=\; \frac{-\sqrt{3}\pi i}{108}.$$

3.

Let

$$\text{☻} := \int_C \frac{z^2e^z}{2z+i}\,dz,$$

where C is the negatively-oriented unit circle in $\mathbb{C}$. Using Cauchy's integral formula with $f(z) = z^2e^z$ and $a = -\frac{i}{2}$, we have $f(-\frac{i}{2}) = \frac{1}{2\pi i}\oint_C \frac{z^2e^z}{z+\frac{i}{2}}\,dz$. Since $f(-\frac{i}{2}) = -\frac{1}{4}e^{-\frac{i}{2}}$, we must have

$$\oint_C \frac{z^2e^z}{z+\frac{i}{2}}\,dz \;=\; 2\pi i(\frac{-e^{-\frac{i}{2}}}{4}) \;=\; -\frac{\pi}{2}ie^{-i/2}.$$

Remembering the orientation of C, we get

$$\text{☻} \;=\; -\frac{1}{2}\oint_C \frac{z^2e^z}{z+\frac{i}{2}}\,dz \;=\; \frac{\pi}{4}ie^{-i/2}.$$

Final answer:

$$\text{☻} \;=\; \frac{\pi}{4}e^{(\pi-1)i/2} \;=\; \frac{\pi}{4}\sin\frac{1}{2} + \frac{\pi}{4}\cos\frac{1}{2}i.$$

Chapter 14, Section 4, Problem 3.

Let

$$f(z) \;=\; \frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right).$$

If $0 < |z| < 1$, then

$$\begin{aligned}
f(z) = &\ \tfrac{1}{z}\left(\tfrac{-\frac{1}{2}}{1-\frac{z}{2}} + \tfrac{1}{1-z}\right) \\
= &\ \tfrac{1}{z}\left(-\tfrac{1}{2}\frac{1}{1-\frac{z}{2}} + \frac{1}{1-z}\right) \\
= &\ \tfrac{1}{z}\left(-\tfrac{1}{2}\left[1 + \tfrac{z}{2} + \left(\tfrac{z}{2}\right)^2 + \left(\tfrac{z}{2}\right)^3 + \cdots\right] + \left[1 + z + z^2 + z^3 + \cdots\right]\right) \\
= &\ \tfrac{1}{z}\left(-\tfrac{1}{2}\left[1 + \tfrac{z}{2} + \tfrac{z^2}{4} + \tfrac{z^3}{8} + \cdots\right] + \left[1 + z + z^2 + z^3 + \cdots\right]\right) \\
= &\ \tfrac{1}{z}\left(\left[-\tfrac{1}{2} - \tfrac{z}{4} - \tfrac{z^2}{8} - \tfrac{z^3}{16} + \cdots\right] + \left[1 + z + z^2 + z^3 + \cdots\right]\right) \\
= &\ \tfrac{1}{2z} + \tfrac{3}{4} + \tfrac{7z}{8} + \tfrac{15z^2}{16} + \cdots .
\end{aligned}$$

So the residue of f at the origin is $1/2$.
If $1 < |z| < 2$, then

$$\begin{aligned}
f(z) &= \tfrac{1}{z}\left(\tfrac{-\frac{1}{2}}{1-\frac{z}{2}} - \tfrac{\frac{1}{z}}{1-\frac{1}{z}}\right) \\
&= \cdots - \tfrac{1}{z^3} - \tfrac{1}{z^2} - \tfrac{1}{2z} - \tfrac{1}{4} - \tfrac{z}{8} - \tfrac{z^2}{16} - \tfrac{z^3}{32} - \cdots .
\end{aligned}$$

If $|z| > 2$, then

$$\begin{aligned}
f(z) &= \tfrac{1}{z}\left(\tfrac{\frac{1}{z}}{1-\frac{2}{z}} - \tfrac{\frac{1}{z}}{1-\frac{1}{z}}\right) = \tfrac{1}{z^2}\left(\tfrac{1}{1-\frac{2}{z}} - \tfrac{1}{1-\frac{1}{z}}\right) \\
&= \cdots + \tfrac{7}{z^5} + \tfrac{3}{z^4} + \tfrac{1}{z^3}.
\end{aligned}$$

Chapter 14, Section 4, Problem 10d.

When z is near π,

$$\cos\left(\tfrac{\pi}{z-\pi}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{\pi}{z-\pi}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}(z-\pi)^{-(2n+1)}.$$

So, π is an essential singularity of $\cos\left(\frac{\pi}{z-\pi}\right)$.