Section 1

(1) \[ v = \frac{x^2}{x^2 + y^2} = \frac{1}{1 + \frac{y^2}{x^2}}. \]

\[ \frac{\partial u}{\partial x} = \frac{-1}{(1 + \frac{y^2}{x^2})^2} \left( \frac{-2y^2}{x^3} \right) \]

\[ \frac{\partial u}{\partial y} = \frac{-1}{(1 + \frac{y^2}{x^2})^2} \left( \frac{2y}{x^2} \right). \]

(2) \[ v = e^x \cos y. \]

(a) \[ \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} e^x \cos y \right) = \frac{\partial}{\partial x} (-e^y \sin y) = -e^y \sin y \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} e^x \cos y \right) = \frac{\partial}{\partial y} e^x \cos y = -e^y \sin y \]

(b) \[ \frac{\partial^2}{\partial x^2} e^x \cos y + \frac{\partial^2}{\partial y^2} e^x \cos y \]

\[ = e^x \cos y + e^x (-\cos y) = 0. \]

Section 4

(2) Letting \( y = \sqrt{x} \), we have \( dy = \frac{dx}{2\sqrt{x}} \).

So \( dy \approx x + \Delta x \).

Taking \( x = n \) and \( \Delta x = a \), \( \sqrt{n+a} - \sqrt{n} \approx \frac{a}{2\sqrt{n}} \).
In the specific example, we have
\[ \sqrt{10^{26} + 5} - \sqrt{10^{26}} \approx \frac{5}{2 \sqrt{10^{26}}} = 2.5 \times 10^{-13}. \]

Section 5

(1) \[ z = xe^{-y}, \quad x = \cosh t, \quad y = \cos t. \]

\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \]
\[ = e^{-y} \cdot \sinh t + (-xe^{-y}) \cdot (-\sin t) \]
\[ = e^{-y} (\sinh t + x\sin t). \]

(7) \[ c = \sin (a-b), \quad b = ae^{2a}. \]

\[ \frac{dc}{da} = \frac{\partial c}{\partial a} \frac{da}{da} + \frac{\partial c}{\partial b} \frac{db}{da} \]
\[ = \cos (a-b) + (-\cos(a-b)) \cdot (e^{2a} + a2e^{2a}) \]

Section 7

(2) \[ P = \gamma \cos t, \quad \gamma \sin t - 2te^{-t} = 0. \]

There are 3 variables and 2 equations, so only one degree of freedom. Thus \( P \) is a function of \( t \) and \( \frac{dP}{dt} \) makes sense.
Taking total differentials:

\[ dp = \frac{\partial p}{\partial r} \, dr + \frac{\partial p}{\partial t} \, dt \]

\[ = \cos t \, dr + (-r \sin t) \, dt \]

and

\[ \theta = \frac{\partial}{\partial r} (rs \sin t - 2te^r) \, dr + \frac{\partial}{\partial t} (rs \sin t - 2te^r) \, dt \]

\[ = (s \sin t - 2te^r) \, dr + (-r \cos t - 2e^r) \, dt \]

Substituting \( dr = \frac{-r \cos t - 2e^r}{s \sin t - 2te^r} \, dt \) into the first equation, we get

\[ dp = -\cos t \left( \frac{r \cos t - 2e^r}{s \sin t - 2te^r} \right) \, dt - rs \sin t \, dt \]

So

\[ \frac{dp}{dt} = -\cos t \left( \frac{r \cos t - 2e^r}{s \sin t - 2te^r} \right) - rs \sin t \]

\[ = -r \cos^2 t - 2e^r \cos t - rs \sin^2 t + 2te^r \sin t \]

\[ = 2te^r \sin t - 2e^r \cos t - \frac{1}{s \sin t - 2te^r} \]
(7) \( x = r \cos \theta, \ y = r \sin \theta. \)

(a) Since \( y \) is an explicit function of \( r, \ \theta, \)
\[
\left( \frac{\partial y}{\partial \theta} \right)_r = r \cos \theta \quad \text{is easy.}
\]

(b) For \( \left( \frac{\partial y}{\partial \theta} \right)_x \), we take differentials:
\[
\begin{align*}
\frac{dx}{d\theta} &= \cos \theta \frac{dr}{d\theta} - r \sin \theta \frac{d\theta}{d\theta} \\
\frac{dy}{d\theta} &= \sin \theta \frac{dr}{d\theta} + r \cos \theta \frac{d\theta}{d\theta}
\end{align*}
\]

Since \( x \) is fixed, we set \( dx = 0 \), which yields (for fixed \( x \)):
\[
\cos \theta \frac{dr}{d\theta} = r \sin \theta \frac{d\theta}{d\theta}
\]

Substituting the expression for \( dr \) into (2):
\[
\frac{dy}{dx} = \sin \theta \left( \frac{r \sin \theta \frac{d\theta}{d\theta}}{r \cos \theta} \right) + r \cos \theta \frac{d\theta}{d\theta}
\]
\[
= \frac{r \sin^2 \theta \frac{d\theta}{d\theta} + r \cos^2 \theta \frac{d\theta}{d\theta}}{r \cos \theta}
\]
\[
= \frac{r \frac{d\theta}{d\theta}}{r \cos \theta}
\]

Thus, \( \left( \frac{\partial y}{\partial \theta} \right)_x = \frac{1}{\cos \theta}. \)

(c) Dividing our original equations gives
\[
\frac{x}{y} = \frac{\cos \theta}{\sin \theta} \implies \theta = \tan^{-1} \left( \frac{y}{x} \right).
\]

So \( \left( \frac{\partial \theta}{\partial y} \right)_x = \frac{1}{1 + \left( \frac{y}{x} \right)^2} \cdot \frac{1}{x} \)
On the other hand, eqn (2) - tanθ × eqn (1) is:

\[ dy = \tanθ \, dx = (r \cosθ + r \sin^2θ) \, dθ \]

So, \( \frac{\partial θ}{\partial y} \bigg|_x = \frac{1}{r \cosθ + r \sin^2θ} = \frac{\cosθ}{r} \).

This is the same as what we got from the first method because

\[ \frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{r \cosθ}{r^2} = \frac{\cosθ}{r} \].

\( \frac{\partial θ}{\partial y} \) and \( \frac{\partial y}{\partial θ} \) are reciprocals only when the variable being held constant is the same.

(14) \[ v = x^2 + y^2 + xyz \], \[ x^4 + y^4 + z^4 = 2x^2y^2z^2 + 10 \].

Taking differentials:

\[ dv = (2x + yz) \, dx + (2y + xz) \, dy + xyz \, dz \]

\[ 4x^3 \, dx + 4y^3 \, dy + 4z^3 \, dz = 4xyz^2 \, dx + 4yz^2 \, dy + 4xz^2 \, dz \].
Since $z$ is held constant, we are interested in the case where $dz=0$, and the equations simplify to:

\[ dw = (2x+yz)dx + (2y+xz)dy \]

\[ 4x^3dx + 4y^3dy = 4x^2y^2dx + 4yx^2z^2dy \]

At the point $(x,y,z) = (2,1,1)$, this becomes:

\[ dw = 5dx + 4dy \]

\[ 24dx + 4dy = 8dx + 16dy \iff 16dx = 12dy \]

So \[ dw = 5dx + 4 \left( \frac{16}{12} \right) dx = \frac{31}{3} dx \]

So \[ \left( \frac{\partial w}{\partial x} \right)_z (2,1,1) = \frac{31}{3} \]

\[ (18) \quad m = a + b, \quad n = a^2 + b^2. \quad \text{Want:} \quad \left( \frac{\partial b}{\partial m} \right)_n, \quad \left( \frac{\partial m}{\partial b} \right)_a. \]

Take differentials:

\[ dm = da + db \]

\[ dn = 2a da + 2b db \]

When $n$ is held constant:

\[ dm = da + db \]

\[ 0 = 2a da + 2b db \]

\[ \Rightarrow da = -\frac{b}{a} db. \]

Substituting:

\[ dm = \left( 1 - \frac{b}{a} \right) db \]

So \[ \left( \frac{\partial b}{\partial m} \right)_n = \frac{1}{1 - \frac{b}{a}} \]

When $a$ is held constant:

\[ dm = db \]

\[ dn = 2b db \]

So \[ \left( \frac{\partial m}{\partial b} \right)_a = 1. \]
\[ w = f(ax + by) \]. Let \( z = ax + by \).

We have:
\[
\begin{align*}
\frac{dw}{dz} &= \frac{df}{dz} (ax + by) \\
&= a \frac{df}{dz} dx + b \frac{df}{dz} dy.
\end{align*}
\]

Thus:
\[
\begin{align*}
\frac{\partial w}{\partial x} &= a \frac{df}{dz} \\
\frac{\partial w}{\partial y} &= b \frac{df}{dz}.
\end{align*}
\]

So:
\[
\begin{align*}
b \frac{\partial w}{\partial z} - a \frac{\partial w}{\partial y} &= b a \frac{df}{dz} - ab \frac{df}{dz} = 0.
\end{align*}
\]

\[ u = f(x - ct) + g(x + ct) \]. We are interested in:
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f(x - ct)}{\partial x^2} + \frac{\partial^2 g(x + ct)}{\partial x^2}.
\]

Applying the conclusion of the previous question to \( f \) and \( g \), we get:

\[
\begin{align*}
-c \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial t} \\
\frac{\partial g}{\partial x} &= \frac{\partial g}{\partial t}
\end{align*}
\]

Applying \( \frac{\partial}{\partial x} \) again gives:
\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= -\frac{1}{c^2} \frac{\partial^2 f}{\partial x \partial t} \\
\frac{\partial^2 g}{\partial x^2} &= \frac{1}{c} \frac{\partial^2 g}{\partial x \partial t}
\end{align*}
\]

So:
\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{c} \frac{\partial^2 g}{\partial x \partial t} - \frac{1}{c} \frac{\partial^2 f}{\partial x \partial t}.
\]
On the other hand, differentiating \( f \) with respect to \( t \) gives:

\[
-c \frac{\partial^2 f}{\partial t^2} = \frac{\partial f}{\partial t} \quad \text{and} \quad c \frac{\partial^2 g}{\partial t^2} = \frac{\partial g}{\partial t}
\]

Adding these, we find that

\[
\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 g}{\partial t^2}
\]

\[
\quad = c \frac{\partial^2 g}{\partial t \partial x} - c \frac{\partial^2 f}{\partial t \partial x}
\]

Assuming continuity of second derivatives, we have

\[
\frac{\partial^2 f}{\partial t \partial x} = \frac{\partial^2 f}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^2 g}{\partial t \partial x} = \frac{\partial^2 g}{\partial x \partial t}.
\]

Thus, combining (1) and (2) gives

\[
\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2}, \quad \text{as desired.}
\]
(26) \( f(x, y, z) = 0 \), \( g(x, y, z) = 0 \).

Taking total differentials:

\[
\begin{align*}
0 &= df = f_x \, dx + f_y \, dy + f_z \, dz \quad \text{(1)} \\
0 &= dg = g_x \, dx + g_y \, dy + g_z \, dz. \quad \text{(2)}
\end{align*}
\]

We eliminate \( dz \) by subtracting \( \frac{f_z}{g_z} \) \times \text{(2)} \) from \( \text{(1)} \):

\[
0 = \left( f_x - \frac{f_z}{g_z} g_x \right) dx + \left( f_y - \frac{f_z}{g_z} g_y \right) dy
\]

Which yields the formula:

\[
\frac{dy}{dx} = -\frac{f_x - \frac{f_z}{g_z} g_x}{f_y - \frac{f_z}{g_z} g_y}.
\]

---

Section 8

(6) \( f(x, y) = x^3 - y^3 - 2xy + 2 \).

The partial derivatives are \( f_x = 3x^2 - 2y \) \( f_y = -3y^2 - 2x \).

These are zero at the critical points, so

\[
\begin{align*}
3x^2 &= 2y \\
-3y^2 &= 2x
\end{align*}
\]

\[
\begin{align*}
3 \left( \frac{-3}{2} y^2 \right) &= 2y \\
\downarrow
\end{align*}
\]

\[
\frac{27}{8} y^2 = 1 \Rightarrow y = \frac{2}{3}.
\]
So \( x = -\frac{3}{2} y^2 = -\frac{3}{2} \left( \frac{2}{3} \right)^2 = -\frac{2}{3} \).

At the point \((-\frac{2}{3}, \frac{2}{3})\) we have
\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 6x = -4 \\
\frac{\partial^2 f}{\partial y^2} &= -6y = -4 \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = -2
\end{align*}
\]

So \( \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} = 4^2 > (\frac{\partial^2 f}{\partial x \partial y})^2 \)

and this point is a \underline{maximum}.

\textbf{Section 9}

(1) The perimeter is \( P(s, l, \theta) = 4s + 2l \)

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & l \\
& & \\
& & \\
& & \\
& & s \\
& & \\
& & \theta \\
& & \\
\end{array}
\]

the base of each triangle is \( 2s \cos \theta \), and the height is \( s \cdot \sin \theta \), so

the total area is
\[
\begin{align*}
A(s, l, \theta) &= 2sl \cos \theta + 2s^2 \sin \theta \cos \theta \\
&= 2sl \cos \theta + s^2 \sin 2\theta.
\end{align*}
\]

At the optimum, we must have \( \nabla A = \lambda \nabla P \) so :
\[
\frac{\partial A}{\partial s} = \lambda \frac{\partial P}{\partial s} \Rightarrow 2l \cos \theta + 2s \sin 2\theta = 4\lambda \quad (1)
\]
\[
\frac{\partial A}{\partial l} = \lambda \frac{\partial P}{\partial l} \Rightarrow 2s \cos \theta = 2\lambda \quad (2)
\]
\[
\frac{\partial A}{\partial \theta} = \lambda \frac{\partial P}{\partial \theta} \Rightarrow -2sl \sin \theta + 2s^2 \cos 2\theta = 0 \quad (3)
\]

(2) and (3) simplify to:

\[
\lambda = s \cos \theta
\]

\[
l = \frac{s \cos 2\theta}{\sin \theta}
\]

Substituting these in (1), we get:

\[
2s \cos 2\theta \cos \theta + 2s \sin 2\theta \sin \theta = 4s \cos \theta
\]

Since \( s \neq 0 \) we cancel from both sides to get:

\[
\cos 2\theta \cos \theta + \sin 2\theta \sin \theta = 2 \sin \theta \cos \theta
\]

Or:

\[
\cos \theta (\cos 2\theta + 2 \sin^2 \theta) = 2 \sin \theta \cos \theta
\]

So either \( \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2} \) (impossible) or

\[
\cos 2\theta + 2 \sin^2 \theta = 2 \sin \theta
\]

\[
\Rightarrow 1 = 2 \sin \theta \Rightarrow \theta = \frac{\pi}{6}
\]

Thus, the optimal solution has \( \theta = \frac{\pi}{6} \).
You can plug this back into the equations to solve for $l$ and $s$ (I will skip it here).

(4) See the modern solutions, #5.

It turns out that in the modern solutions I solved a different problem:

I used the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \)

instead of \( \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \).

The method is the same, even though the numbers are slightly different. Sorry about that!
(11) We want:

\[
\begin{align*}
\min & \quad x^2 + y^2 + z^2 = f(x, y, z) \\
\text{subject to} & \quad 2x + y - z = \varphi_1(x, y, z) = 1 \quad (1) \\
& \quad x - y + z = \varphi_2(x, y, z) = 2 \quad (2)
\end{align*}
\]

The method of Lagrange multipliers for two constraints tells us that at the minimum:

\[
\nabla f = \lambda_1 \nabla \varphi_1 + \lambda_2 \nabla \varphi_2
\]

for some \( \lambda_1, \lambda_2 \).

This implies that:

\[
\begin{align*}
2x &= \lambda_1 (2) + \lambda_2 (1) = 2\lambda_1 + \lambda_2 \quad (3) \\
2y &= \lambda_1 (1) + \lambda_2 (-1) = \lambda_1 - \lambda_2 \quad (4) \\
2z &= \lambda_1 (-1) + \lambda_2 (1) = -\lambda_1 + \lambda_2 \quad (5)
\end{align*}
\]

Substituting these in (1), (2), we have

\[
2\lambda_1 + \lambda_2 + \frac{\lambda_1 - \lambda_2}{2} - (-\frac{\lambda_1 + \lambda_2}{2}) = 1
\]

and

\[
2\lambda_1 + \lambda_2 - \frac{(\lambda_1 - \lambda_2)}{2} + \frac{(\lambda_1 + \lambda_2)}{2} = 2
\]

which after gathering terms becomes:

\[
3\lambda_1 = 1 \quad , \quad \frac{3\lambda_2}{2} = 2 \quad \Rightarrow \quad \lambda_1 = \frac{1}{3} \\
\lambda_2 = \frac{4}{3}
\]


Substituting back in (3), (4), (5), we get

\[ x = \lambda + \frac{\lambda^2}{2} = \frac{1}{3} + \frac{1}{2} \left( \frac{4}{3} \right) = \frac{1}{2} \]

\[ y = \frac{\lambda - \lambda^2}{2} = \frac{1}{3} - \frac{1}{3} \left( \frac{1}{3} \right) = -\frac{1}{2} \]

\[ z = -\lambda + \lambda^2 = -\frac{1}{3} + \frac{1}{3} = \frac{1}{2} \]

Section 10

(1) we want:

\[ \min x^2 + y^2 = f(x, y) \]

subject to \[ x^2 - y^2 = 1 \rightarrow \phi(x, y) \]

Using Lagrange Multipliers, we have

\[ 2x = \lambda \cdot 2x \implies x = \lambda x \]

\[ 2y = \lambda(-2y) \implies y = -\lambda y \]

So either \[ x = 0, \lambda = -1 \]

or \[ y = 0, \lambda = 1 \]

or \[ x = y = \lambda = 0 \]

The first and last cases are impossible since \[ x^2 - y^2 = 1 \], so we must have \[ y = 0 \] and \[ \lambda = 1 \], where \[ x = \pm 1 \]. So the two minima are \((1, 0)\) and \((-1, 0)\).
I think the book put this in this section because another way to solve it is to use the second equation to make the substitution

\[ y^2 = x^2 - 1 \]

which gives the unconstrained problem

\[ \min \ x^2 + x^2 - 1 = 2x^2 - 1 \]

with the implicit constraint \( z^2 - 1 \geq 0 \) since \( y^2 \geq 0 \)

Using calculus here gives the solution

\[ 4x - 1 = 0 \implies x = \frac{1}{4} \]

which violates the implicit constraint. Thus, we have to check the boundary values \( x^2 = 1 \)

\[ \implies x = \pm 1 \]

and sure enough these are the minima we derived using Lagrange multipliers.
(8) We want: \[ \text{max } T(x, y, z) = xyz \]
\[ \text{subj. } x^2 + y^2 + z^2 = \Phi(x, y, z) = 12. \]

This is almost identical to #5 on the midterm as well as Section 9 #4 on this HW (with the numbers changed), and can be solved using the same Lagrange multiplier method.

The only difference is that we do not require \( x, y, z \) to be positive since they are no longer dimensions of a box, but coordinates in space.

(11) The (unconstrained) max/min temperature satisfies:

\[ \frac{\partial T}{\partial x} = 0 \implies 4x - 2 = 0 \implies (x, y) = (2, 0) \]
\[ \frac{\partial T}{\partial y} = 0 \implies -6y = 0 \]

which is outside the box

So any optimum must lie on the boundary.
Let us consider the 4 restrictions:

\[ T = 2(x^2 - 3y^2 - 2(1) + 10 \]
\[ = -3y^2 + 10 \]

Any interior min/max satisfies: Correction
\[ -6y + 10 = 0 \quad \Rightarrow \quad y = \frac{10}{6} > 1 \]
So outside the box.
Thus, we only need to consider the corners. \( T(1,0) = 12 \)

\[ x = -1 \]
\[ T = 2(-1)^2 - 3y^2 + 2 + 10 \]
\[ = -3y^2 + 14 \]
\[ \frac{\partial T}{\partial y} = -6y = 0 \]
So \( (-1,0) \) is a critical point.
\[ T(-1,0) = 14 \]

\[ y = 1 \]
\[ T = 2x^2 - 3 - 2x + 10 \]
\[ \frac{\partial T}{\partial x} = 4x - 2 \]
\[ \Rightarrow \quad x = \frac{3}{2} \]
So there is a critical point at \((\frac{3}{2}, 1)\)
\[ T(\frac{3}{2},1) = \frac{13}{2} \]

\[ x = -1 \]
\[ T = 2(-1)^2 - 3y^2 + 2 + 10 \]
\[ \frac{\partial T}{\partial y} = -6y = 0 \]
So there is a critical point at \((\frac{3}{2}, -1)\).
\[ T(\frac{3}{2},-1) = \frac{13}{2} \]
The values at the corners are:

\[ T(1, 1) = 2 - 3 - 2 + 10 = 7 \]
\[ T(1, -1) = -3 + 10 = 7 \]
\[ T(-1, 1) = -3 + 14 = 11 \]
\[ T(-1, -1) = -3 + 14 = 11 \]

So the minimum is at \( (1, -1) \) with \( T = 7 \).

and the maximum is at \( (-1, 0) \) with \( T = 14 \).