Math 121A Spring 2015, Homework 10

*Due April 20 at 5pm in my office, or in class*

We will denote the Fourier transform of \( f : \mathbb{R} \to \mathbb{C} \) as
\[
\hat{f}(\alpha) = (\mathcal{F} f)(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} \, dx
\]
and the inverse Fourier transform of \( \hat{f} : \mathbb{R} \to \mathbb{C} \) as:
\[
(\mathcal{F}^{-1} \hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{i\alpha x} \, d\alpha.
\]

Do not worry about convergence of integrals for this homework — assume all functions are nice enough so that the relevant Fourier transforms exist and the Fourier inversion theorem \( \mathcal{F}^{-1} \mathcal{F} f = f \) works.


2. Boas 12.6, 12.10, 12.12, 12.24 (read pages 381-832), 12.27, 12.34 (read page 384).

3. (a) Show that if \( f \) is real-valued, then \( \hat{f}(-\alpha) = \overline{\hat{f}(\alpha)} \).

(b) Show that if \( f \) is even then \( \hat{f} \) is even, and if \( f \) is odd then \( \hat{f} \) is odd.

What do (a) and (b) together say about the Fourier transform of a real even function? A real odd function?

(c) Let \( f^{rev}(x) = f(-x) \) be the reversal/reflection of \( f \). Show that
\[
\mathcal{F}^{-1} f = \mathcal{F} f^{rev}.
\]

For this problem, you have to ignore the ‘type’ of \( f \) (i.e., whether it is a function of \( x \) or of \( \alpha \)) and treat \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) simply as operators which take a function and spit out another function.

4. (a) Show that \( f \ast g = g \ast f \).

(b) Use 1c and the fact that
\[
\mathcal{F}(f \ast g) = \sqrt{2\pi} \mathcal{F} f \cdot \mathcal{F} g
\]
to show that
\[
\mathcal{F}(f \cdot g) = \frac{1}{\sqrt{2\pi}} \mathcal{F} f \ast \mathcal{F} g,
\]
i.e., that the Fourier transform turns multiplication into convolution.

5. Let
\[
I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx.
\]

Express the integral
\[
I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} \, dx \, dy
\]
in polar coordinates, and use this expression to conclude that \( I = 1 \).
6. Use the Fourier transform to solve the heat equation on an infinite line:

\[
\frac{d}{dt} u(x, t) = \frac{d^2}{dx^2} u(x, t)
\]

with initial conditions

\[ u(x, 0) = e^{-x^2/2}. \]

Describe the distribution of heat at time \( t \) in words. What happens as \( t \to \infty \)?

7. For a function \( f : \mathbb{R} \to \mathbb{C} \) with \( f(0) \neq 0 \), define the rectangular width of \( f \) to be the width of a rectangle with height \( f(0) \) and area equal to that under the graph of \( f(x) \), i.e.

\[
W_f = \frac{1}{f(0)} \int_{-\infty}^{\infty} f(x) dx.
\]

Show that the product of the rectangular width of a function and that of its Fourier transform is \( 2\pi \), i.e.

\[ W_f \cdot W_{Ff} = 2\pi. \]

Thus, if a function has small rectangular width, its Fourier transform must have large rectangular width.

This can be viewed as a baby version of the Uncertainty Principle, which says that the product of the variance of \( f \) and the variance of \( Ff \) is large. (This is the same as the uncertainty principle in physics, since position and momentum are Fourier transforms of each other).