4/13/15

In the last lecture we introduced the convolution $f*g$ of two functions f and g, motivated by the question: what is the inverse Fourier transform of the product $\hat{f}\hat{g}$?

This is easy for the sum: $\mathcal{F}^{-1}(\hat{f}+\hat{g}) = f+g$, but for the product it turned out to be

$$(\mathcal{F}^{-1}\hat{f}\hat{g})(x) = \frac{(f*g)(x)}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)\,dy .$$

This is a very useful operation (even outside the context of Fourier transforms), and corresponds to computing a sort of "moving weighted average" of f with respect to "weights" given by g (though in general g may not be a probability density or even positive).

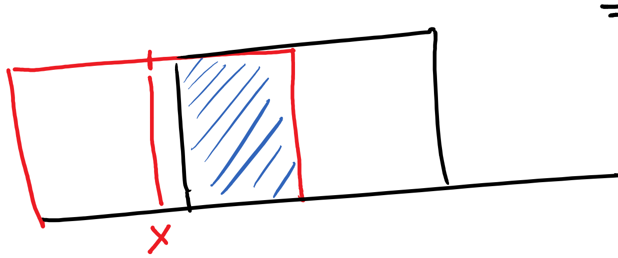
Here are some examples to illustrate this:

eg: Let $\Pi(x) = \begin{cases} 1 & -1/2 < x < 1/2 \\ 0 & \text{otherwise} \end{cases}$ be the "box" function.

Note that the area of the box is $\underline{1}$.

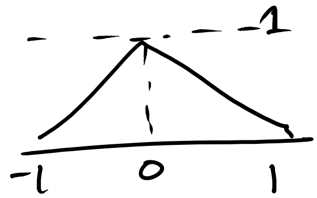
Then: $\Pi * \Pi(x) = \displaystyle\int_{-\infty}^{\infty} \Pi(y)\,\Pi(x-y)\,dy$

$$= \int_{x-1/2}^{x+1/2} \Pi(y)\,dy$$



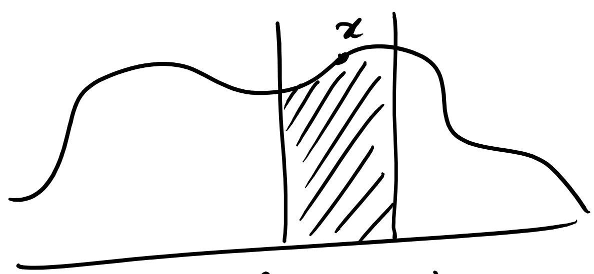
$$= 1-|x|$$

is the area of the intersection of a box centered at zero (the $\Pi(y)$ term) and a "flipped" box centered at x (the $\Pi(x-y)$ term, which because Π is even is the same as $\Pi(y-x)$).

So the graph of $\Pi * \Pi$ looks like a triangle: 

In general, the convolution operation may be remembered "flipping and dragging" one function against the other.

When one of the functions is an indicator function of an interval (i.e. a translation/scaling of $\Pi(x)$), the convolution $(f*\Pi)(x)$ is just the average value of $f(y)$ in an interval of width 1 centered at x, and corresponds exactly to the area under the graph of f restricted to this interval:



$(f*\Pi)(x)$

Quite often, one of the functions g is an even probability density (i.e. $\int_{-\infty}^{\infty} g(x) = 1$, $g(x) \geq 0$, $g(-x) = g(x)$) and in this case $(f*g)(x)$ simply replaces the value of f at x by its average with respect to the distribution g.

A very common function to convolve with is the Gaussian:

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

This occurs in the solution of the heat equation, and also for instance in image processing — when photoshop blurs an image, it essentially convolves with a Gaussian.

In general, the convolution of two functions inherits the smoothness properties of both the functions:

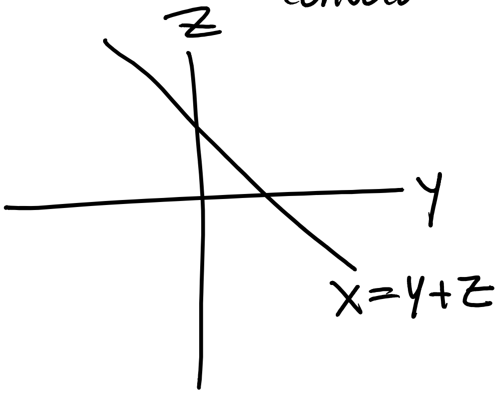
$$(f * g)' = f' * g = f * g'$$

Optional:
Convolution also shows up in Probability. Suppose I have two independent random variables Y and Z with densities $\psi(y)$ and $\omega(z)$,

i.e.
$$\mathbb{P}[a \le Y \le b] = \int_a^b \psi(y)\, dy$$

$$\mathbb{P}[a \le Z \le b] = \int_a^b \omega(z)\, dz$$

Now, the density function of the <u>sum</u> $Y+Z$ is given by the convolution



$$\mathbb{P}[a \le Y+Z \le b] = \int_a^b \int_{-\infty}^{\infty} \psi(y)\, \omega(x-y)\, dy\, dx$$

$$= \int_a^b (\psi * \omega)(x)\, dx.$$

Thus, $(y * v)(x)$ represents all the different ways to get $Y+Z = x$.

The Central limit theorem (which is beyond the scope of this course) says that if you add up many independent random variables, the (appropriately normalized) sum looks like a Gaussian.

This is a consequence of the fact that convolving <u>any</u> (suitably nice) function with itself many times brings it closer and closer to a Gaussian:

$$f * f * f * \cdots f * \xrightarrow[\text{appropriately scaled.}]{} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Try this with the Π function on a computer; the results after just 4-5 convolutions are quite striking.

See Question 21 on Boas p389 for an outline of how to derive the Poisson summation formula.