4/1/15

Fourier Series, 2

Last time, we ended with Fourier's original application of using the exponential Fourier series to solve the heat equation on a circle.

The important points were:

1. There are three equivalent ways to think of a periodic function with period $2\pi$:
   - as a periodic function on $\mathbb{R}$
   - as an arbitrary function on the unit circle, denoted $S^1$
   - as a function on any interval of length $2\pi$, with the endpoints identified, e.g. $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$, $f'(\pi) = f'(\pi)$, etc.

These are called periodic boundary conditions.

The length of the interval/ circumference of the circle are important, since we usually use that $\sin(nx), \cos(nx), e^{inx}$ are periodic with respect to this length, i.e. $e^{in\pi} = e^{i(-n\pi)}$, etc.

Everything can be made to work for intervals of another length $2L$, such as $[-L,L]$, if we replace $\sin(nx)$ by $\sin\left(\frac{2\pi}{L}x\right)$, $e^{inx}$ by $e^{\frac{2\pi ix}{L}}$, etc.
Periogicity allows us to use integration by parts to "move" the \( \frac{d^2}{dx^2} \) operator from one term to a product to the other:

\[
\int_{-\pi}^{\pi} \frac{d^2}{dx^2} e^{-inx} \, dx = e^{-inx} \frac{du}{dx} \bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{du}{dx} \frac{de^{-inx}}{dx} \, dx
\]

\[
= - \left[ \frac{de^{-inx}}{dx} \cdot v \bigg|_{-\pi}^{b} - \int_{-\pi}^{\pi} v \cdot \frac{d^2 e^{-inx}}{dx^2} \, dx \right]
\]

\[
= \int_{-\pi}^{\pi} v \cdot \frac{d^2}{dx^2} e^{-inx} \, dx.
\]

Note: This argument would have worked if we replaced \( e^{-inx} \)
by any periodic function with the same period:

\[
\int_{-\pi}^{\pi} \frac{d^2}{dx^2} f \cdot g \, dx = \int_{-\pi}^{\pi} f \cdot \frac{d^2}{dx^2} g \, dx
\]
for any \( f, g \) periodic with period \( 2\pi \) or satisfying periodic boundary conditions on \([-\pi, \pi]\).

Thus, it is more a property of the \( \frac{d^2}{dx^2} \) operator and boundary conditions than the particular function in this problem.
\[ \frac{d^2}{dx^2} e^{inx} = (in)^2 e^{inx} = -n^2 e^{inx}, \]

i.e., the exponential functions are eigenfunctions of the \( \frac{d^2}{dx^2} \) operator. This is a very special property which \( \sin(nx) \) and \( \cos(nx) \) also share, but which no other functions have.

4) The functions \( e^{inx} \) are "orthogonal" in that
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}. \]

This allows us to compute the Fourier coefficients in \( f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} \) just by computing integrals.

**Notation:** We will denote the \( n \)-th Fourier coefficient \( \hat{f}(n) \) of \( f \) by \( \hat{f}(n) \). This is handy when there are multiple functions involved.

Thus, we will write \( f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} \)

where \( \hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx. \)
The most coherent way to understand what is going on is in terms of inner product spaces.

The relevant space here is \( L^2[-\pi, \pi] \) "ell two"

\[
= \left\{ f: [-\pi, \pi] \to \mathbb{C} : \int_{-\pi}^{\pi} |f(x)|^2 \, dx < \infty \right\}
\]

with the inner product

\[
\langle f | g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx.
\]

You can check that this is a vector space and that

\( \langle f | g \rangle \) satisfies all the axioms of an inner product (see week 3 notes). The inner product allows us to define a norm

\[
\|f\| = \sqrt{\langle f | f \rangle}
\]

and together they behave just like the Euclidean dot product/norm in \( \mathbb{R}^n \). In particular, they satisfy:

1. The triangle inequality: \( \|f+g\| \leq \|f\| + \|g\| \)
2. Pythagoras theorem: \( \|f+g\|^2 = \|f\|^2 + \|g\|^2 \) when \( \langle f | g \rangle = 0 \)

The point is that this allows us to use our geometric intuition and concepts like projection, length, orthogonality when thinking about \( L^2 \).
In this notation, the exponential functions are actually orthonormal:

\[ \langle e^{inx} | e^{imx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{imx} \, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \]

And the Fourier coefficients are inner products

\[ \hat{f}(n) = \langle e^{inx} | f \rangle. \]

The punch line is that the exponentials are an orthonormal basis:

**Theorem:** If \( f \in L^2[-\pi,\pi] \) then the partial sums

\[ S_N(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \sum_{n=-N}^{N} \langle e^{inx} | f \rangle e^{inx} \]

converge in mean square to \( f \):

\[ \| f - S_N \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - S_N(x)|^2 \, dx \to 0 \]

as \( N \to \infty \).

This means that the vectors \((e^{inx})\) are getting closer and closer to \( f \) in norm.

In terms of the function, it means that the "area" of the squared difference goes to zero:
This is an average notion of convergence which does not tell us that \( S_N(x) \to f(x) \) if we plug in any particular point \( x \).

So it is weaker than the kind of convergence we studied in the first two weeks.

The advantages of considering it are:

- It works under extremely general and natural conditions (square-integrability is much weaker than being differentiable, continuous, etc.)

- It comes equipped with a powerful geometric intuition. (In hw9 you will show that the \( N \)th partial sum of the Fourier series is the least squares approximation to \( f \) using \( e^{-iN\pi x} \ldots e^{iN\pi x} \).)

- It is adequate for many physical applications (such as in quantum mechanics) where we are more interested in integrals of the square of a function (which correspond to probability amplitudes or energy) than in evaluating the function at particular points. In fact, \( L^2 \) convergence is also called “convergence in energy”.
One important consequence of the theorem is:

**Parseval's theorem:** If \( f \in L^2[-\pi, \pi] \) then

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \left\| f \right\|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 = \sum_{n \in \mathbb{Z}} |\langle e^{inx}, f \rangle|^2
\]

i.e. the squared length of a vector (which is in this case a function) is equal to the sum of squares of its inner products with an orthonormal basis.

This is exactly what happens in \( \mathbb{R}^n \).

It is very powerful because it allows one to evaluate integrals using series and vice versa (see HW9).

In the next lecture, we will consider a stronger notion of convergence which requires more restrictive assumptions on \( f \).
Coupled Oscillator Example

\[ \begin{align*}
V_1(t) & \quad V_2(t) \\
\end{align*} \]

positions at time \( t \): \( V(t) \in \mathbb{R}^2 \)

\[ \frac{d^2 V}{dt^2} = AV \]

\( A \) is hermitian

\[ \Rightarrow \text{eigenvectors} \]
\[ A b_1 = \lambda_1 b_1, \quad A b_2 = \lambda_2 b_2 \]

form an orthonormal basis.

Expand \( V(t) = C_1(t)b_1 + C_2(t)b_2 \)

In eigenbasis, diff eq decouples

\[ \frac{d^2 C_i(t)}{dt^2} = \lambda_i C_i(t) \]

Easy to solve scalar differential eqns.

\[ \square \]