3/30/15

Fourier Series

- A function $f: \mathbb{R} \to \mathbb{C}$ is periodic with period 2ℓ if $f(x+2\ell) = f(x)$ for all $x \in \mathbb{R}$.

Such a function is completely determined by its values on any closed interval of length 2ℓ.

- Examples: $\sin(x)$ has period 2π
  $\sin(2x)$ has period π
  $\sin(nx), \cos(nx)$ have period $\frac{2\pi}{n}$.
  e^{ix} has period 2π

Square Wave $f(x) = \begin{cases} -1 & x \in (-\pi, 0) \\ 1 & x \in (0, \pi) \\ 0 & x \in 0, \pi \end{cases}$ has period 2π

Sawtooth $f(x) = x \quad x \in (-\pi, \pi]$ has period 2π

The last two functions are much rougher than the first few.

Deep Fact: Every* periodic function can be decomposed as an infinite sum of sines and cosines / exponentials.

(Trigonometric) Fourier Expansion:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

(Exponential) Fourier Expansion:

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

The main questions are:

Q1) <u>Which</u> functions have fourier expansions?

Q2) <u>How</u> do you find the expansion?

Q3) <u>Why</u> is this useful?

Q4) <u>Why</u> is it possible? What is special about sin/cos/exp?

Let's start with Q2.

Assume f is periodic with period 2π and can be expanded as $f(x) = \sum_n c_n e^{inx}$.

We will just think of f as being a function on $[-\pi, \pi]$.

<u>Key observation</u>:

$$\int_{-\pi}^{\pi} e^{inx} = \begin{cases} 2\pi & \text{if } n=0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

(easy exercise)

This allows us to isolate the coefficients C_k:

$$\int_{-\pi}^{\pi} e^{-ikx} f(x) = \sum_n C_n \int_{-\pi}^{\pi} e^{-ikx} e^{inx} dx$$

$$= 2\pi C_k \quad \text{since} \quad \int_{-\pi}^{\pi} e^{i(n-k)x} = 0 \text{ for } n \neq k.$$

Thus, we have $\boxed{C_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx}$.

The reason it is easy to compute the coeffs is that the functions e^{inx} are "orthogonal" in that the integrals $\int e^{inx} e^{-imx} dx$ vanish for $n \neq m$. We will develop this more formally in the next lecture

A similar thing can be done for $\sin(nx)$ and $\cos(nx)$.
It is described in the book.

The two kinds of series really describe the same thing. The relationship between them is revealed by:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{a_0}{2} + \sum_n \frac{a_n}{2}\left(e^{inx} + e^{-inx}\right) + \frac{b_n}{2i}\left(e^{inx} - e^{inx}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2}\right)e^{inx} + \left(\frac{a_n + ib_n}{2}\right)e^{-inx}$$

We have not shown this, but Fourier series are unique.
Thus, we must have

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}.$$

One consequence of this is that $\overline{c_n} = -c_n$ for all n iff f is real-valued.

<u>Example</u>: Explicit calculation of Fourier coefficients for the square wave function.

<u>Example</u>: Fourier in developed this in the 1800's to study the <u>heat equation</u> on a circle.

Let S^1 denote the circle of radius 1, with circumference 2π, and consider a distribution of heat $f(x)$, $f: S^1 \to \mathbb{R}$.
Let $v(x,t)$ be the distribution at time t, with initial condition $v(x,0) = f(x)$. We are interested in how $v(x,t)$ evolves with time, according to the heat equation

$$v_t = \alpha v_{xx}$$ where α is a constant depending on the material.

We can think of $v(x,t)$ as being periodic on the real line with period 2π.

Fourier considered what would happen if the solution could be written as a sum of exponentials:

$$v(x,t) = \sum_{n=\infty}^{\infty} c_n(t)\, e^{inx}$$

for some coefficients $c_n(t)$ depending on time (i.e., for every fixed t these coefficients give us a function $v(x,t)$ of x).

This way of describing functions turns out to be very powerful in this context, and "decouples" the partial differential equation into many scalar ordinary differential equations, just as in the spring examples in week 3.

In particular, solving the problem is equivalent to solving for the coefficients $c_n(t)$.

We know $c_n(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$ are just the Fourier coefficients of x.

If we try to compute their derivatives, we find

$$\frac{d}{dt}c_n(t) = \frac{d}{dt}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(x,t)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d}{dt}v(x,t)\, e^{-inx}\, dx$$

(optional:
Someone asked me when it is ok to exchange derivatives & integrals.
One sufficient condition, given by Leibniz, is that the partial derivatives $v_x(x,t)$ and $v_t(x,t)$ are both continuous.
If we write the derivative as a limit, this boils down to a question of exchanging limits and integrals, which is answered using the bounded convergence theorem.)

$$= \frac{1}{\alpha 2\pi}\int_{-\pi}^{\pi} \alpha \frac{d^2}{dx^2} v(x,t)\, e^{-inx}\, dx$$

We now move the $\frac{d^2}{dx^2}$ to the exponential using integration by parts and the fact that v is periodic in x:

$$I = \int_{-\pi}^{\pi} \frac{d^2}{dx^2} v(x,t)\, e^{-inx}\, dx = e^{-inx}\frac{d}{dx}v(x,t)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(-in)e^{-inx}\frac{dv}{dx}\, dx$$

Key point: The first term vanishes because

$$e^{-in\pi}\frac{dv}{dx}(\pi,t) = e^{-in(-\pi)}\frac{dv}{dx}(-\pi,t)$$

due to periodicity (because $-\pi$ and π are the same point)

Thus, we can apply integration by parts again to obtain

$$I = in \int_{-\pi}^{\pi} \frac{du}{dx} e^{-inx} dx = in \left[e^{-inx} v \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (-in) e^{-inx} v \, dx \right]$$

$$= (in)^2 \int_{-\pi}^{\pi} e^{-inx} v(x,t) \, dx$$

Thus, we have

$$c_n'(t) = \frac{\alpha}{2\pi} (in)^2 \int_{-\pi}^{\pi} e^{-inx} v(x,t) \, \cancel{dt} \to dx$$

$$= -n^2 \alpha \, c_n(t).$$

Notice that the $\frac{d^2}{dx^2}$ has been replaced by multiplication! This is because the exponentials e^{inx} are eigenvectors of the $\frac{d}{dx}$ operator.

This is now a simple scalar ODE in time, with solution $c_n(t) = \exp(-\alpha n^2 t) c_n(0)$.

Thus, the general solution is

$$v(x,t) = \sum_{n \in \mathbb{Z}} c_n(t) e^{inx}$$

$$= \sum_{n \in \mathbb{Z}} \exp(-\alpha^2 n t) c_n(0) e^{inx},$$

and we see that the coefficients on the modes e^{inx} decay exponentially, independently, with time.