Math 121A Spring 2015, Sample Final Exam

May 4, 2015

1 Sample Final

1. Consider the 2π-periodic function defined by

$$f(x) = \begin{cases} -1 & -\pi < x \leq 0 \\ 1 & 0 < x \leq \pi \end{cases}$$

   Expand f in a Fourier series and use Parseval's theorem to find the sum of

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

2. Compute the (exponential) Fourier transform of

$$f(x) = \frac{e^{ix}}{1+x^2}.$$

3. Derive an expression for the function $(f * f)^2$, where

$$f(x) = e^{-x^2/2}.$$

4. Using the Laplace transform method or otherwise, solve the differential equation

$$y''(t) + \omega^2 y(t) = \delta(t - t_0) + \delta(t - 2t_0) \qquad y(0) = y'(0) = 0 \quad t \geq 0,$$

   describing a harmonic oscillator, initially at rest, struck twice by a hammer at times t_0 and $2t_0$ with $t_0 > 0$. For which values of t_0 is the oscillator eventually at rest?

5. Find the Green's function for the boundary value problem:

$$y''(x) + y(x) = f(x) \qquad y(0) = 0 \quad y'(\pi) = 0 \qquad x \in [0, \pi].$$

   Use the Green's function to find a solution of:

$$y''(x) + y = x \qquad y(0) = 0 \quad y'(\pi) = 0.$$

6. Evaluate the integral

$$\int_{|z|=3} z^3 \exp(1/z^2) dz,$$

   where the contour is oriented positively (counterclockwise).

7. Using a keyhole contour or otherwise, evaluate the integral

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2} dx.$$

8. Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} n(n-1)x^{n-2}.$$

   To what function does the series converge within this radius?

9. Determine whether the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   is diagonalizable. If it is, find an eigenbasis, if not, explain why.

10. If $z = xe^{-y}$, $x = \cosh(t)$, and $y = \cos(s)$, find $\partial z/\partial s$ and $\partial z/\partial t$.

2 Practice Problems

1. Suppose $f(x)$ is a $2\pi-$periodic function, $f'(x)$ exists and is continuous everywhere, and $\int_{-\pi}^{\pi} f(x)dx = 0$. Use Parseval's theorem to show that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

2. Calculate the (exponential) Fourier series expansion for the function

$$f(x) = \frac{1}{1+\lambda e^{ix}}, \quad \lambda < 1.$$

3. Find the Laplace transform of $g(t) = tf(t)$ in terms of the Laplace transform of $f(t)$ (Hint: differentiate the definition of the Laplace transform). Use this to calculate the Laplace transform of

$$t^2 e^{-\pi t} \sin(t).$$

4. Calculate

$$\cos(\theta) + \cos(2\theta) + \ldots + \cos(n\theta).$$

5. Find the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{1-x^2} dx.$$

6. Compute the exponential Fourier transform of

$$f(x) = e^{-|x|}\cos(x).$$

7. Use the Laplace transform and the Bromwich inversion integral to solve the initial value problem

$$\frac{d^4 y(t)}{dt^4} + y(t) = 1 \quad y(0) = 1, y'(0) = y''(0) = y'''(0) = 0 \quad t \geq 0.$$

8. Find the Green's function for the boundary value problem:

$$y''(x) - y(x) = f(x) \qquad y'(0) = y'(\pi) = 0.$$

9. Find the Green's function for the boundary value problem:

$$y''(x) - y(x) = f(x) \qquad y(0) = y(\pi) \quad y'(0) = y'(\pi).$$