Intro to Complex Numbers

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1 Arithmetic

Historically, complex numbers were invented because there are quadratic equations, such as

$$x^2 + 1 = 0$$

without real solutions. The solution to this particular equation is called the imaginary number i: $i^2 = -1$,[1] and one way to define the set of complex numbers is as the set of all expressions of type $x + iy$ where x and y are real. This is denoted by $\mathbb{C}$.

What is important is the operations on this set. Arithmetic on complex numbers is defined precisely as if $x + iy$ was a polynomial in the variable i, except with the important twist that $i^2 = -1$. In particular, for complex numbers $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$, we have

$$z_1 + z_2 = z_2 + z_1 := x_1 + iy_1 + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 z_2 = z_2 z_1 := (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + i(y_1x_2 + x_1y_2) + i^2 y_1y_2 = x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2).$$

What is less obvious is that every nonzero complex number also has an inverse; in particular, if $z = x + iy$ then the number

$$z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

(note that the denominators are real) satisfies $zz^{-1} = z^{-1}z = 1$ (and this formula can be derived by solving a system of linear equations, as we did in class). The existence of an inverse allows us to sensibly define division for a pair of complex numbers, $z_2 \neq 0$, as $z_1/z_2 := z_1 z_2^{-1}$.

If $z = x + iy \in \mathbb{C}$, then $x = \mathrm{Re}(z)$ is called the *real part* of z and $y = \mathrm{Im}(z)$ is called the *imaginary part* of z. Note that all of the real numbers are also complex numbers, with imaginary part equal to zero. The complex numbers also have an additional operation called *conjugation*, which arises from the fact that there is a symmetry between i and $-i$ in the equation $x^2 + 1 = 0$, and there is no reason a priori to prefer one over the other (it is just a matter of convention which we one we choose to call i). The conjugate of $z = x + iy$ is $\overline{z} = x - iy$, i.e., the number obtained by taking the negative of the imaginary part, thereby replacing i with $-i$.

Multiplying any complex number by its conjugate yields a nonnegative real number:

$$z\overline{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2 \geq 0.$$

[1] Actually, there are *two* solutions, i and $-i$; this will come up again on page 2.

This number has a geometric interpretation if we visualize $z = x + iy$ as a two dimensional vector with Cartesian coordinates x and y: it is the squared Euclidean length. Accordingly, the square root of this product is called the *magnitude* of a complex number, denoted

$$|z| = (z\overline{z})^{1/2},$$

where we take the nonnegative square root. This is the generalization of "absolute value" for real numbers, and it is easy to see that the magnitude of a real number is equal to its absolute value.

With the conjugate and magnitude in hand, it is easy to come up with the following compact formula for the inverse:

$$z^{-1} = \frac{\overline{z}}{|z|^2}, \qquad \text{so} \qquad zz^{-1} = \frac{z\overline{z}}{|z|^2} = 1.$$

The conjugate and magnitude behave in the nicest possible ways with respect to the arithmetic operations; in particular, it is easy to check that:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} \qquad \overline{z^{-1}} = \overline{z}^{-1} \quad \overline{(z_1/z_2)} = \overline{z_1}/\overline{z_2}.$$

$$|z_1 z_2| = |z_1||z_2| \qquad |z^{-1}| = 1/|z| \qquad |z_1/z_2| = |z_1|/|z_2| \qquad |z_1 + z_2| \leq |z_1| + |z_2|.$$

Note that the last one is an *inequality.* It is called the *triangle inequality* and follows from the fact that complex number addition is identical to vector addition and corresponds to 'completing a triangle', whence the length of the sum (the third side) is at most the sum of the lengths of the vectors. Recall that the same inequality is also true for real numbers.

It is tempting to think of $\mathbb{C}$ as being the same thing in $\mathbb{R}^2$, since we can identify every point with the vector (x, y). While it is useful to visualize it this way, the two settings not the same, for at least two reasons: (1) It is meaningful to multiply and divide complex numbers, but this is not so for vectors (2) The complex plane has two distinguished axes: the line $\text{Im}(z) = 0$, called the real axis, and the line $\text{Re}(z) = 0$, called the imaginary axis. The operations of multiplication and division behave *very differently* for these two axes: for instance, as we shall see in a moment, multiplying a number by i 'rotates' it by $\pi/2$, whereas multiplying a number by 1 leaves it unchanged. On the other hand, the standard axes do not have any particular significance in $\mathbb{R}^2$, and everything would be the same if we chose any other orthonormal basis.

2 Polar Coordinates

It is easy to visualize what happens when you add two complex numbers, since this is the same as vector addition, which is simple in Cartesian coordinates. To visualize complex multiplication and division (i.e., inversion), it is better to use polar coordinates. If we make the change of variables $x = r\cos\theta, y = r\sin\theta$, with $r \geq 0$, we can write every complex number as

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

Observe that the magnitude is just what we expect:

$$|z| = |r||\cos\theta + i\sin\theta| = r(\cos^2\theta + \sin^2\theta) = r.$$

In this form, it becomes very clear what complex multiplication is doing:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right) \\ &= r_1 r_2\left(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2)\right), \end{aligned}$$

by high school trigonometry. Thus, multiplying two complex numbers is equivalent to multiplying their magnitudes and adding their angles. In particular, multiplying by i corresponds to rotating by $\pi/2$, since $i = \cos(\pi/2) + i\sin(\pi/2)$. This gives a 'geometric' explanation of why $i^2 = -1$: two counterclockwise rotations by $\pi/2$ is the same thing as a reflection. The existence of this geometric interpretation is remarkable, since we started with a purely algebraic situation (the quadratic equation $x^2 + 1 = 0$). Note that this rotational feature of multiplication is completely absent in the multiplication of real numbers, which are one dimensional and only have room for scaling/reflection.

In general, the angle θ is called the *argument* of z, denoted $\arg(z)$. It is important to note that because sin and cos are periodic with period 2π, the argument is not unique, and so it is not actually a function, but rather can take infinitely many values for each input z. For instance, $\arg(1)$ could be $0, 2\pi, 4\pi, \ldots$, and in general, if θ is an argument of z, then so is any $\theta + 2\pi k$ for integer k. Luckily, the values that it takes are all just shifts of each other by 2π, so they are easy to understand. This is sometimes called a 'multivalued function', and it is nothing to be scared about. It is similar to what happens when you try to invert a real function which is not one to one; for instance $y = \sqrt{x}$, the inverse of $x = y^2$, always has two values for nonzero x (a positive one and a negative one). In that situation, we adopt the convention that $\sqrt{x}$ means the nonnegative square root.

We will do a similar thing here, but more explicitly: we use the notation $\mathrm{Arg}(z)$ to denote the value of the argument of z that lies in the interval $(-\pi, \pi]$. This is called the *principal branch* of the argument. It has the advantage of being an actual function, and the disadvantage of behaving a bit strangely when z is close to the negative real axis (i.e., has argument $-\pi$). In particular, $\mathrm{Arg}(-1+\epsilon i)$ is very close to π for small ϵ, but for the nearby point $-1 - \epsilon i$, we have $\mathrm{Arg}(-1 - \epsilon i) \approx -\pi$, so the function is discontinuous near the negative real axis.

3 The Complex Exponential

So far, our switch to polar coordinates is rather superficial, in that it is just a different way of parameterizing Cartesian coordinates — something we could have just as easily have done in $\mathbb{R}^2$, and as such not really a feature of the complex numbers. However, there is a much deeper connection between the polar and cartesian forms, given by the complex exponential function.

As in the real case, the complex exponential is defined by the infinite series:

$$e^z = 1 + z + z^2/2! + z^3/3! + \ldots,$$

which is absolutely convergent for all $z \in \mathbb{C}$ by the ratio test (see the book for a satisfactory discussion of complex series and disks of convergence). Note that there is no mention of trigonometry or geometry in this definition, which relies only on complex arithmetic and the ability to take limits of partial sums.

But now, if we substitute a pure imaginary $z = i\theta$, we find that

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + (i\theta)^2/2! + (i\theta)^3/3! + (i\theta)^4/4! + \dots \\ &= 1 + (i\theta) - \theta^2/2! - i\theta^3/3! + \theta^4/4! + \dots \\ &= \left(1 - \theta^2/2! + \theta^4/4! + \dots\right) + i\left(\theta - \theta^3/3! + \theta^5/5! + \dots\right) \\ &\quad \text{collecting real and imaginary parts (rearrangement is ok by absolute convergence)} \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

This is known as Euler's identity. Among its beautiful consequences are the fact that $e^{i\pi} = -1$, by taking $\theta = \pi$. (In fact, this is a very reasonable way to *define* π.) As far as parameterizing complex numbers goes, it tells us that

$$r(\cos\theta + i\sin\theta) = re^{i\theta},$$

which is a much more substantial statement than just taking $x = r\cos\theta$ and $y = r\sin\theta$, since the right hand side is an infinite series (with very special analytic properties, which we will exploit) and the left hand side is a finite sum. This is the polar form that we shall use.

One of the important properties of the exponential, which may be checked by multiplying the absolutely convergent series term by term, is that

$$e^{z_1}e^{z_2} = e^{z_1+z_2},$$

for every complex z_1 and z_2. This also tells us that for arbitrary $z = x + iy$, we have $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$.

Multiplication, inversion, and conjugation (but not addition) all become exceedingly transparent in the polar form:

$$(r_1e^{i\theta_1})(r_2e^{i\theta_2}) = r_1r_2e^{i(\theta_1+\theta_2)} \qquad (re^{i\theta})^{-1} = r^{-1}e^{-i\theta} \qquad \overline{re^{i\theta}} = re^{-i\theta},$$

and this makes such calculations very easy.

One cool application for now is to trigonometric identities. Suppose I want the formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$. By the properties established above, we know that

$$\cos(3\theta) + i\sin(3\theta) = e^{i(3\theta)} = (e^{i\theta})^3 = (\cos(\theta) + i\sin(\theta))^3.$$

The right hand side can be expanded using the binomial theorem to give:

$$\cos^3\theta + i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + 3i^3\sin^3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta),$$

from which the formula can be read off easily by comparing real parts. We can derive formulas like this for any multiple of θ completely mechanically, without any need for trigonometric insight. The point is that the algebra of complex numbers implicitly 'knows trigonometry' and is doing all the work for us.