The Bromwich Inversion Integral for Rational Functions

In class, we showed that for a function $f(t)$ of exponential order σ (and with finitely many discontinuities, maxima, and minima on every finite interval), the Laplace transform

$$F(p) = \int_0^\infty f(t)\, e^{-pt}\, dt \quad , \quad \mathrm{Re}(p) > \sigma$$

may be inverted as a complex line integral:

we will use z instead of p from now on

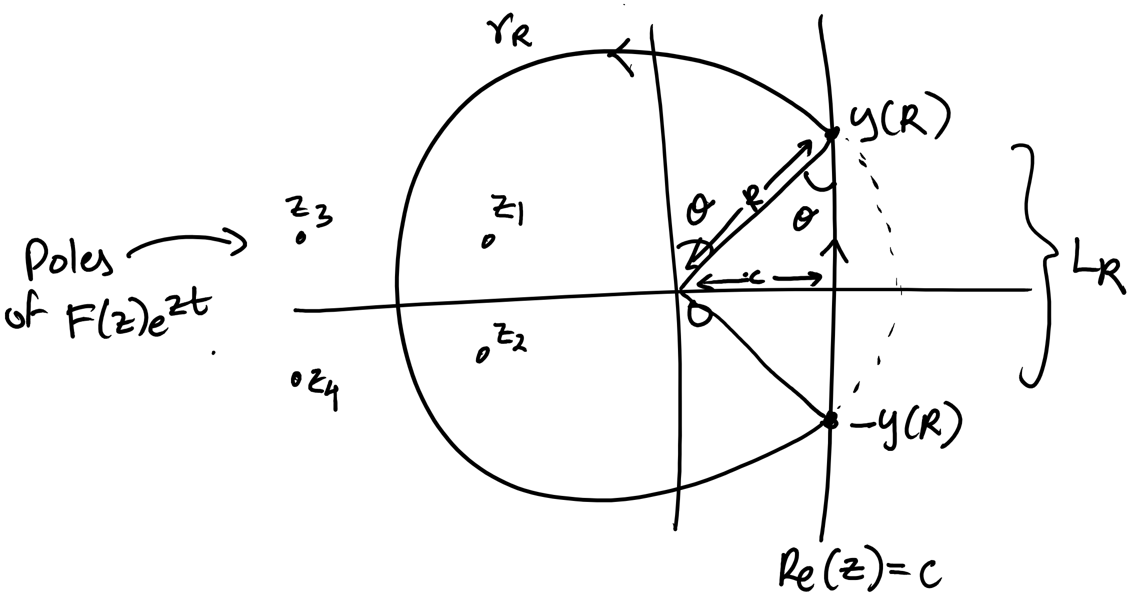
$$f(t) = \frac{1}{2\pi i} \lim_{y \to \infty} \int_{c-iy}^{c+iy} F(z) e^{tz}\, dz =: \lim_{y\to\infty} I_y$$

where $c > \sigma$.

In general this is evaluated by closing the contour and using the Residue theorem; the details may be more or less complicated depending on the function F.

When $F(z) = \dfrac{P(z)}{Q(z)}$ for some polynomials P, Q with $\deg(P) < \deg(Q)$, the contour is quite simple and the application of the residue theorem is enlightening.

Note that though the integral defining $F(z)$ only converges for $z > \sigma$, the function $F(z) = \dfrac{P(z)}{Q(z)}$ is defined and analytic everywhere except the zeros of Q.



We consider a circle of radius R, which intersects the vertical line of interest $Re(z) = c$ at two points $y(R)$ and $-y(R)$; call the line segment between them L_R. The part of the circle to the left of the line may be parameterized as

$$\gamma_R(s) = R e^{i(\frac{\pi}{2}+s)}, \quad s \in [-\theta, \pi+\theta]$$

where $\theta \doteq \sin^{-1}\left(\frac{c}{R}\right)$ is the angle shown in the figure.

Notice that as $R \to \infty$, $\sin\theta = \frac{c}{R} \to 0$ so $\theta \to 0$.

We will consider the positively oriented closed contour $C_R = L_R + \gamma_R$.

The Residue theorem tells us that

$$\int_{C_R} F(z) e^{zt} dz = 2\pi i \sum_{j=1}^{K} Res(z_j)$$

where z_j are the poles of $F(z)e^{zt}$ inside C_R.

As $R \to \infty$, the sum includes all the poles of $F(z)e^{zt}$ (since all the poles must be to the left of the line, because $F(z) = \int_0^\infty f(t)e^{-zt}\,dt$ converges to the right of the line).

On the other hand, we will show that

$$\lim_{R\to\infty} \int_{\gamma_R} F(z)e^{zt}\,dz = 0$$

i.e. the circular part of the integral vanishes as $R \to \infty$. This will be done using Jordan's lemma:

Since the parametrization may be written as:

$$z = \gamma_R(s) = Re^{i(s+\frac{\pi}{2})} = e^{i\frac{\pi}{2}}Re^{is} = iRe^{is}$$

$$dz = i^2 Re^{is}ds = -Re^{is}\,ds \quad , \text{ we have}$$

$$\int_{\gamma_R} F(z)e^{zt}\,dz = -\int_{-\theta}^{\pi+\theta} \underbrace{\frac{P(iRe^{is})}{Q(iRe^{is})}\, e^{itRe^{is}}\, Re^{is}}_{\star}\,ds \,.$$

Since $t>0$ this is exactly the setting for Jordan's lemma, except for the angle $\theta \neq 0$.

It may be checked using a triangle inequality argument that

$$\lim_{R\to\infty} \int_{-\theta}^{\pi+\theta} \star\,ds = \lim_{R\to\infty} \int_0^\pi \star\,ds$$

since $\theta \to 0$ as $R \to \infty$.

We conclude that

$$\lim_{R\to\infty} \int_{L_R} F(z)\, e^{zt}\, dz = \lim_{R\to\infty} \int_{C_R} F(z) e^{zt}\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(z_j)$$

So $$\boxed{f(t) = \sum_{j=1}^{k} \operatorname{Res}\left(F(z)e^{zt}, z_j\right)}$$

i.e. the inverse Laplace transform is precisely the sum of the residues at the poles of $F(z)e^{zt}$, which are the same as the poles of $F(z)$ since e^{zt} is entire.

When f is the solution of a differential equation, its qualitative behavior may be read off immediately from the locations of the poles (the real parts control exponential growth or decay, while the imaginary parts determine oscillatory behavior).

(See pages 696–698 of the book for an example)

Note that in this calculation, the integral

$$F(z) = \int_0^\infty f(t) e^{zt} \, dt$$

is only defined in the region $D = \{ \mathrm{Re}(z) > \sigma \}$

but the function $F(z)$ is defined

(and is analytic) everywhere except its singularities,

and we calculated the Bromwich integral by

studying the behavior of F outside D !

[A more general F (not necessarily a rational function), can be similarly extended to a larger region of $\mathbb{C}$ using a process known as analytic continuation.

This is kind of argument plays a crucial role in many questions in analytic number theory, including the famous Riemann Hypothesis.]