Chapter 6

Direct constructions of point source solutions

The principle of constructing solutions due to a distributed source by superposition of point source solutions was discovered by Fourier analysis. Now, it takes on a life of its own. Very often, we can construct point source solutions directly, completely bypassing Fourier analysis. The essential ideas are presented in an "ascending sequence" of prototype examples.

Kicking a particle in goo

Let $v(t)$ be the x velocity of a particle in viscous liquid ("goo"). The particle is subject to some force per unit mass. The x-component as a function of time is denoted by $f(t)$. $v(t)$ satisfies the ODE

$$\dot{v}(t) + \beta v(t) = f(t). \tag{6.1}$$

Here, $\beta > 0$ is the particle's damping coefficient due to its emersion in "goo". We assume the particle is at rest for $t < 0$. The time interval $0 < t < T$ corresponds to a "kick", so $f(t)$ is non-zero only in $0 < t < T$. The impulse per unit mass associated with the kick is

$$I := \int_0^T f(t)dt. \tag{6.2}$$

Let's assume that a finite impulse I is delivered in a very short time, shorter than we can resolve. The "physicists' shorthand" to describe this situation is formal representation $f(t)$ by a Dirac delta function,

$$f(t) = I\delta(t). \tag{6.3}$$

The initial value problem for $v(t)$ is

$$\dot{v}(t) + \mathbf{v}(t) = I\delta(t), \tag{6.4}$$

for all t, subject to $v(t) \equiv 0$ for $t < 0$.

What is $v(t)$ in $t > 0$? In $t > 0$, $\delta(t) \equiv 0$, and the solutions of the ODE (6.4) in $t > 0$ are

$$v(t) = v_0 e^{-\beta t}, \ t > 0. \tag{6.5}$$

The constant v_0 is determined by integrating the ODE (6.4) over $-\varepsilon < t < \varepsilon$. We have

$$v_0 e^{-\beta\varepsilon} + \mathbf{v}_0 \int_0^\varepsilon e^{-\beta t} dt = I.$$

By taking the limit $\varepsilon \to 0$, we find $v_0 = I$, so $v(t) = Ie^{-\beta t}$ in $t > 0$. If the particle is initially at rest, and the impulse is delivered at time $t = t'$, the corresponding *impulse response* is

$$v(t) = \left[\begin{array}{rl} 0, & t < t', \\ Ie^{-\beta(t-t')}, & t > t'. \end{array} \right. \tag{6.6}$$

Figure 6.1 depicts this impulse response.

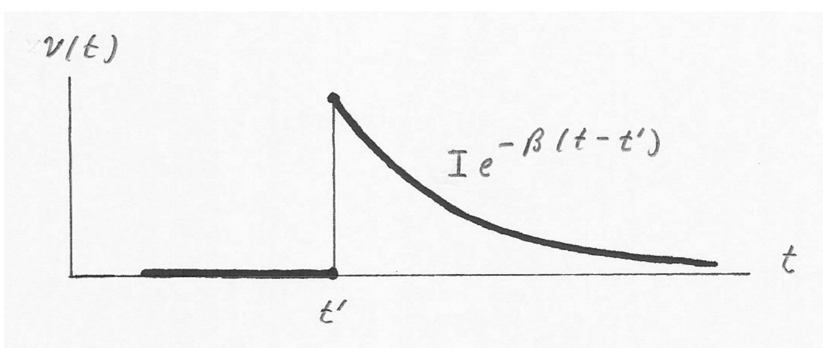

Figure 6.1

Now suppose we have a general, continuous force $f(t) \neq 0$ in $t > 0$. The total impulse delivered in the time interval $(t', t' + dt')$ is $f(t')dt'$ and the contribution of this impulse to $v(t)$ is

$$dv(t) = \left[\begin{array}{rl} 0, & t < t', \\ (f(t')dt')e^{-\beta(t-t')}, & t > t'. \end{array} \right. \tag{6.7}$$

The "total" $v(t)$ due to all the impulses delivered in the time interval $(0, t)$ is the *superposition* of impulse responses

$$v(t) = \int_0^t f(t')e^{-\beta(t-t')}dt'. \tag{6.8}$$

In an exercise, this solution is verified by solving the ODE (6.1) by the integrating factor method.

Kicked harmonic oscillator

$x(t)$ is the displacement of a harmonic oscillator from its rest position. For $t < 0$, it is at rest, with $x(t) \equiv 0$. At $t = 0$, we deliver an impulse I. The formal ODE for $x(t)$ is

$$\ddot{x} + x = I\delta(t). \tag{6.9}$$

Again, what is $x(t)$ in $t > 0$? In $t > 0$, $\delta(t) \equiv 0$ and the general solution for $x(t)$ is linear combinations of $\cos t$ and $\sin t$. Physically, we expect that the impulse at $t = 0$ imparts an initial velocity $\dot{x}(0^+)$, but *no* instantaneous displacement from $x = 0$, so $x(0^+) = 0$. Hence, we have

$$x(t) = v_0 \sin t$$

in $t > 0$. We find the constant v_0 by formal integration of (6.9) over $-\varepsilon < t < \varepsilon$:

$$v_0 \cos \varepsilon + v_0 \int_0^\varepsilon \sin t \; dt = I.$$

In the limit $\varepsilon \to 0$ we find $v_0 = I$. If the impulse is delivered at time $t = t'$, we obtain the harmonic oscillator's impulse response,

$$x(t) = \left[\begin{array}{rl} 0, & t < t'. \\ I \sin(t - t'), & t > t'. \end{array} \right. \tag{6.10}$$

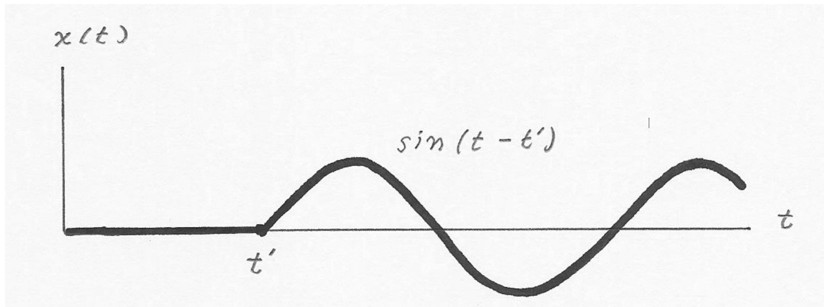

Figure 6.2

Figure 6.2 is the graph of the impulse response (6.10). For a general continuous force $f(t) \neq 0$ in $t > 0$, the solution for $x(t)$ is a superposition of impulse responses analogous to (6.8),

$$x(t) = \int_0^t f(t') \sin(t - t') dt'. \tag{6.11}$$

This proposed solution can be verified by another standard mathematical construction, called "variation of parameters". This is the subject of an exercise.

Walking on water

You've seen little bugs who "walk on water": Their feet, supported by surface tension and hydrostatic pressure, don't break through the surface. We examine a little one-dimensional model: $y(x)$ is a vertical displacement of the water surface from equilibrium. In Figure 6.3, the little up-pointing arrows represent hydrostatic pressure force, which pushes up when the surface is depressed below equilibrium. The big arrows tangential to water surface at $x = x_1, x_2$ represent surface tension helping "hold up" the portion of water surface between $x = x_1$ and $x = x_2$. The vertical downward arrows represent the distributed load due to the bug's foot. The hydrostatic pressure and surface tension forces acting together balance the weight of the foot. The (dimensionless) force balance in $x_1 < x < x_2$ is expressed by

$$y'(x_2) - y'(x_1) - \int_{x_1}^{x_2} y(x) dx = \int_{x_1}^{x_2} f(x) dx. \tag{6.12}$$

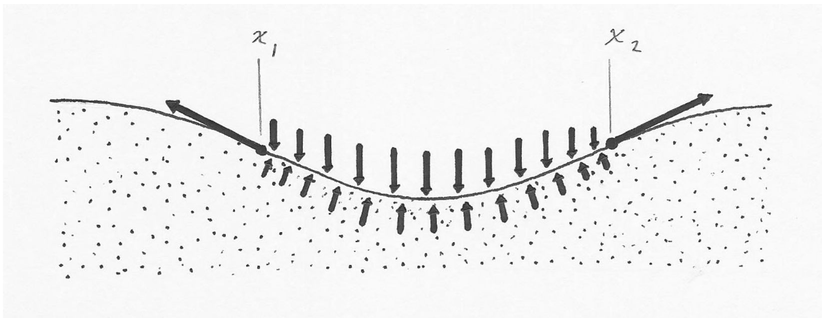

Figure 6.3

In the left-hand side, $y'(x_2) - y'(x_1)$ is the vertical component of the surface tension force, and $-\int_{x_1}^{x_2} y(x)dx$ is the hydrostatic pressure force. Their sum balances the load on $x_1 < x < x_2$ due to the foot. $f(x)$ is the load per unit length in the x-direction, so the total load on $x_1 < x < x_2$ is $\int_{x_1}^{x_2} f(x)dx$ as it appears in the right-hand side of (6.12). From (6.12) we deduce

$$\int_{x_1}^{x_2} (y'' - y - f)dx = 0,$$

presumably for all x_1 and $x_2 > x_1$. Hence, $y(x)$ satisfies the ODE

$$y'' - y = f. \tag{6.13}$$

Now suppose the bug's foot shrinks to very small size, but the total load $F := \int_{-\infty}^{\infty} f(x)dx$ remains finite. "Physics shorthand" represents this situation by $f(x) = F\delta(x)$ and the ODE (6.13) becomes

$$y'' - y = F\delta(x). \tag{6.14}$$

We expect that the water's surface looks like Figure 6.4: A localized "dip", with a "corner" at $x = 0$. This foot is supported by surface tension forces at $x = 0^+, 0^-$. For $x \neq 0$, we have the homogeneous ODE, $y'' - y = 0$, whose elementary solutions are e^x, e^{-x}. Since y should asymptote to zero as $|x| \to \infty$, and is continuous at $x = 0$, the $x \neq 0$ solution for $y(x)$ takes the form

$$y(x) = \left[\begin{array}{ll} y(0)e^{-x}, & x > 0 \\ y(0)e^{x}, & x < 0 \end{array}\right. = y(0)e^{-|x|}. \tag{6.15}$$

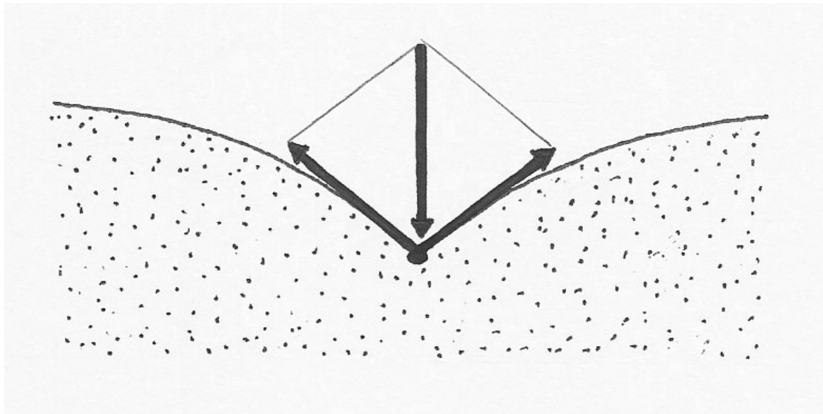

Figure 6.4

The constant $y(0)$ is determined by integration of (6.14) over $-\varepsilon < x < \varepsilon$. This time we get

$$y'(\varepsilon) - y'(-\varepsilon) - \int_{-\varepsilon}^{\varepsilon} y(x)dx = F,$$

and substituting for $y(x)$ from (6.15),

$$-y(0)e^{-\varepsilon} - y(0)e^{-\varepsilon} - y(0)\int_{-\varepsilon}^{\varepsilon} e^{-|x|}dx - F.$$

Taking the limit $\varepsilon \to 0$, we find $y(0) = -\frac{F}{2}$ and the "point load" solution for $y(x)$ is

$$y(x) = -\frac{F}{2}e^{-|x|}. \tag{6.16}$$

The solution for $y(x)$ corresponding to a continuous load per unit length $f(x)$ is another superposition analogous to (6.8), (6.11):

$$y(x) = -\frac{1}{2}\int_{-\infty}^{\infty} f(x')e^{-|x-x'|}dx'. \tag{6.17}$$

You've seen this solution before. In an exercise, you derived it via Fourier analysis.

Radiation condition

Here is a close kindred of the previous example: In the forced wave equation (5.52), take $f(x) = \delta(x)$, corresponding to a "unit point source" at

$x = 0$. In chapter 5, we presented a Fourier analysis of the point source solution,

$$u(x,t) = g(x)e^{i\omega t} \tag{6.18}$$

where $g(x)$ is given by (5.68). Setting $z = i\omega\sqrt{1 - \frac{i\beta}{\omega}}$ (with positive real part), we see that $g(x)$ in (5.68) is

$$g(x) = \frac{1}{2z}e^{-z|x|} \tag{6.19}$$

which looks like a souped up version of the point load solution (6.16). Indeed the direct construction of (6.19) is very much like that of (6.16):

First, $g(x)$ satisfies the ODE

$$g'' + z^2 g = -\delta(x). \tag{6.20}$$

which is (5.54) with $U(x) = g(x)$ and $f(x) = \delta(x)$. In the Fourier analysis, there is no initial mention of boundary conditions on $g(x)$. Rather, it is understood that Fourier integral representations vanish as $|x| \to \infty$, so a zero boundary condition on $g(x)$ at $|x| = \infty$ is *implicitly* assumed. The explicit, *physical* idea of this boundary condition at infinity is clear: Waves go outward from the source. Since they are *damped*, their amplitude vanishes at spatial infinity. The solutions of (6.20) which vanish as $|x| \to \infty$ and are continuous at $x = 0$ take the form

$$g(x) = Ce^{-z|x|}, \tag{6.21}$$

where the constant C is determined by the usual argument: Substitute (6.21) into the ODE (6.20) and integrate over $-\varepsilon < x < \infty$. We find $C = \frac{1}{2z}$, and we're back to (6.19).

Just one more comment before we move on: In the undamped limit $\beta \to 0$,

$$g(x) \to \frac{1}{2i\omega}e^{-i\omega|x|}$$

and the solution (6.18) for the wavefield $u(x,t)$ becomes

$$u(x,t) = \frac{1}{2i\omega}e^{i\omega(t-|x|)}. \tag{6.22}$$

Observe that a right traveling wave fills all of $x > 0$ with *no* attenuation, and similarly, a left traveling wave fills all of $x < 0$. In the absence of damping, we don't have $u \to 0$ as $|x| \to \infty$. What distinguishes the solution (6.22) from all others is that waves are *outgoing* from $x = 0$. If you are a moth stuck on a strand of spider web, and you are flapping away, you are generating *outgoing* waves which inform the "lord of the manor" that "dinner is served". In wave propagation problems, imposing *outgoing* waves from a source is called the *radiation condition*. Even though undamped wave equations are invariant under reversing the direction of time, the solutions (conventionally) regarded as "physically relevant" break the symmetry between past and future, precisely because of the radiation condition.

Periodic impulse response

Recall the chapter 5 Fourier analysis of an RC circuit driven by a time periodic voltage. Given the angular frequency ω, we represented the voltage by $V(\theta := \omega t)$ where $V(\cdot)$ is 2π periodic. The induced periodic current has the integral representation

$$I(\theta) = \frac{V(\theta)}{R} - \frac{1}{R}\int_{-\pi}^{\pi} V(\theta')F(\theta - \theta')d\theta', \tag{6.23}$$

where the 2π periodic kernel $F(\theta)$ is given by the piecewise exponential

$$F(\theta) = \begin{cases} -\dfrac{e^{-\frac{\pi+\theta}{2}}}{2\sinh\frac{\pi}{\Omega}}, & -\pi < \theta < 0, \\ -\dfrac{e^{\frac{\pi-\theta}{2}}}{2\sinh\frac{\pi}{\Omega}}, & 0 < \theta < \pi, \end{cases} \tag{6.24}$$

in the period interval $-\pi < \theta < \pi$. Here, $\Omega := \omega RC$ is the dimensionless frequency.

We present an alternative "real time" analysis leading to (6.23). The starting point is ODE governing the circuit in Figure 5.6. These are

$$\begin{aligned} \omega\frac{dQ}{d\theta} &= I(\theta), \\ Q(\theta) &= C(V(\theta) - RI(\theta)). \end{aligned} \tag{6.25}$$

Here, $Q = Q(\theta)$ is the charge on the capacitor. In the first equation, $\omega\frac{dQ}{d\theta}$ is time derivative of charge, equal to the current. In the second equation,

$V(\theta) - RI(\theta)$ is the voltage drop across capacitor, which is $CQ(\theta)$. Combining equations (6.25) gives an ODE for $Q(\theta)$ with $CV(\theta)$ as a "source":

$$\Omega \frac{dQ}{d\theta} + Q = CV(\theta). \tag{6.26}$$

Given the periodic solution of (6.26) for $Q(\theta)$, the current is

$$I(\theta) = \frac{V(\theta)}{R} - \frac{Q(\theta)}{RC}\,. \tag{6.27}$$

From (6.26), (6.27), we discern the essential problem: Given 2π periodic $V(\theta)$, construct the 2π periodic solution of (6.26) for $Q(\theta)$, and insert it into (6.22) to get the periodic current. It is sufficient to solve (6.26) in the period interval $-\pi < \theta < \pi$, subject to *periodicity condition*

$$Q(-\pi) = Q(\pi). \tag{6.28}$$

We then obtain $Q(\theta)$ for all θ by the obvious periodic extension.

First, take $V(\theta)$ to be a "unit voltage spike", $V(\theta) = \delta(\theta)$, so (6.26) becomes

$$\Omega \frac{dQ}{d\theta} + Q = C\delta(\theta). \tag{6.29}$$

This resembles the "kicked particle ODE" (6.4) from the very first example. Similar to the first example, we expect that $Q(\theta)$ has a jump discontinuity at $\theta = 0$ induced by the voltage spike at $\theta = 0$. The essential difference is that $Q(\theta)$ must satisfy the periodicity condition (6.28). Such solutions take the form

$$Q(\theta) = \begin{bmatrix} qe^{-\frac{\pi+\theta}{\Omega}}, & -\pi < \theta < 0, \\ qe^{\frac{(\pi-\theta)}{\Omega}}, & 0 < \theta < \pi, \end{bmatrix} \tag{6.30}$$

where q is a constant. Its value is determined by the usual argument of integrating the ODE (6.29) over $-\varepsilon < \theta < \varepsilon$ and taking the limit $\varepsilon \to 0$. We find

$$-qe^{\frac{\pi}{\theta}} + qe^{-\frac{\pi}{\Omega}} = C,$$

so

$$q = -\frac{C}{2\sinh\frac{\pi}{\Omega}}\,.$$

Given this value of q, (6.30) is exactly the same as the kernel $F(\theta)$ in (6.24) times $C : Q(Q) = CF(\theta)$. Then $I(\theta)$ in (6.27) becomes

$$I(\theta) = \frac{\delta(\theta)}{R} - \frac{1}{R}F(\theta). \tag{6.31}$$

We pause for a moment of physical insight: Because we are considering periodic voltage and current, the voltage spike $V(\theta) = \delta(\theta)$ in $-\pi < \theta < \pi$ means that the periodic voltage for all θ is the periodic extension

$$V(\theta) = \sum_{-\infty}^{\infty} \delta(\theta - 2\pi n). \tag{6.32}$$

The corresponding current is the "periodic impulse response",

$$I(\theta) = \sum_{-\infty}^{\infty} \frac{\delta(\theta - 2\pi n)}{R} - \frac{1}{R}F(\theta) \tag{6.33}$$

where $F(\theta)$ is the periodic extension of its values in $-\pi < \theta < \pi$ in (6.24).

The current due to general, continuous $V(\theta)$ is constructed by superposition. In the period interval $-\pi < \theta < \pi$, $V(\theta)$ is represented as a formal superposition of unit voltage spikes by the selection identity,

$$V(\theta) = \int_{-\pi}^{\pi} V(\theta')\delta(\theta - \theta')d\theta'.$$

We obtain the current by replacing $\delta(\theta - \theta')$ with the periodic impulse response in (6.33), with $\theta - \theta'$ replacing θ. For θ in $-\pi < \theta < \pi$, this gives

$$I(\theta) = \frac{V(\theta)}{R} - \frac{1}{R}\int_{-\pi}^{\pi} V(\theta')F(\theta - \theta')d\theta',$$

which reproduces the result (6.23) of Fourier analysis.