Chapter 5

Fourier series and transforms

Physical wavefields are often constructed from superpositions of complex exponential traveling waves,

$$e^{i(kx-\omega(k)t)}. \tag{5.1}$$

Here the *wavenumber* k ranges over a set D of real numbers. The function $\omega(k)$ is called the *dispersion relation*, which is dictated by the physics of the waves. If D is some *countable* set of real numbers, the superposition takes the form of a linear combination

$$\psi(x,t) = \sum_{k \text{ in } D} \hat{f}(k) e^{i(kx-\omega(k)t)}. \tag{5.2}$$

The coefficients $\hat{f}(k)$ are complex numbers, one for each k in D. Physical wavefields can be represented as the real or imaginary parts of (5.2). For instance, in the chapter 4 example of "beats in spacetime", D consists of two distinct wavenumbers k_1 and $k_2 \neq k_1$ and $\hat{f}(k_1) = \hat{f}(k_2) = 1$. Suppose the wavefield (5.1) is periodic in x, with period of say, 2π. Then D consists of all the integers, $\cdots -2, -1, 0, 1, 2, \dots$. Let $f(x) := \psi(x,0)$ denote the 2π periodic initial condition at $t = 0$. Then (5.2) at $t = 0$ becomes

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx}. \tag{5.3}$$

It is natural to ask if there are coefficients $\hat{f}(k)$ so an arbitrary 2π periodic function $f(x)$ is represented by the superposition (5.3). For a large class of $f(x)$ the answer is "yes" and the superposition on the right-hand side is called the *Fourier series* of $f(x)$. Suppose $f(x)$ is real: By use of the Euler formula $e^{ikx} = \cos kx + i \sin kx$, and the even and odd symmetries of $\cos kx$, $\sin kx$, we can rewrite (5.3) as a linear combination of $\cos kx$, $k = 0, 1, 2, \dots$ and $\sin kx$, $k = 0, 1, 2, \dots$,

$$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} k_x \sin kx, \tag{5.4}$$

where a_k and b_k are real. The derivation of this *real Fourier series* from (5.3) is presented as an exercise. In practice, the complex exponential Fourier series (5.3) is best for the *analysis* of periodic solutions to ODE and PDE, and we obtain concrete *presentations* of the solutions by conversion to real Fourier series (5.4).

If the set D of wavenumber is the whole real line $-\infty < k < \infty$, the natural generalization of the discrete sum (5.2) is the *integral*

$$\psi(x, t) = \int_{-\infty}^{\infty} \hat{f}(k) e^{i(kx - \omega(k)t)} dk. \tag{5.5}$$

Here, $\hat{f}(k)$ is a function defined on $-\infty < k < \infty$. Let $f(x) := \psi(x, 0)$ be the initial values of ψ at $t = 0$. Then (5.5) reduces to

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk. \tag{5.6}$$

As for the case of the Fourier series (5.3), we ask: Is there $\hat{f}(k)$ so that the integral on the right-hand side of (5.6) represents a given $f(x)$? For a broad class of $f(x)$ with sufficiently rapid decay of $|f(x)|$ to zero as $|x| \to \infty$, the answer is yes, and $\hat{f}(k)$ in (5.6) is called the *Fourier transform* of $f(x)$.

We've introduced Fourier series and transforms in the context of wave propagation. More generally, Fourier series and transforms are excellent tools for analysis of solutions to various ODE and PDE initial and boundary value problems. So lets go straight to work on the main ideas.

Fourier series

A most striking example of Fourier series comes from the summation formula (1.17):

$$\cos\theta + \cos 3\theta + \dots \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}\,. \tag{5.7}$$

We recall that the derivation of (5.7) can be done by elementary geometry. Integrating (5.7) over $0 < \theta < x$, we find

$$\begin{aligned} \sin x + \frac{1}{3}\sin 3x + \dots \frac{1}{2n-1}\sin(2n-1)x &= \int_0^x \frac{\sin 2n\theta}{2\sin\theta} d\theta \\ &= \int_0^{2nx} \frac{\sin\vartheta}{4n\sin\frac{\vartheta}{2n}} d\vartheta. \end{aligned} \tag{5.8}$$

The last equality comes from changing the variable of integration to $\vartheta = 2n\theta$. Figure 5.1 is the graph of the integrand $\frac{\sin\vartheta}{4n\sin\frac{\vartheta}{2n}}$ as a function of ϑ in

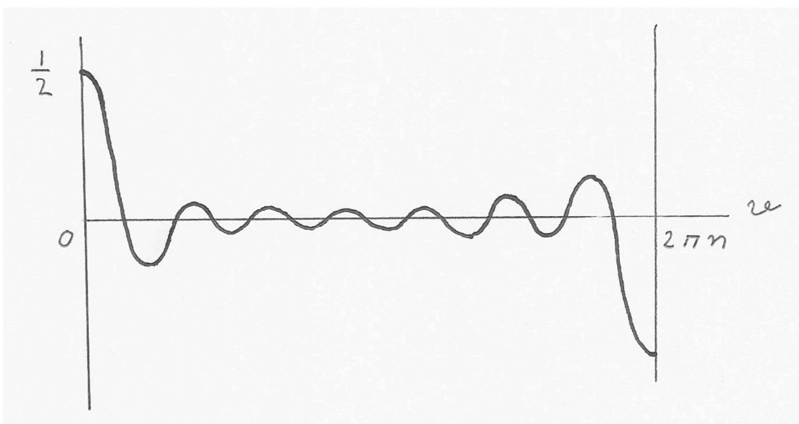

Figure 5.1

$0 \le \vartheta \le 2\pi n$. The graph looks like "teeth in the mouth of a snaggle-tooth cat". Take fixed x in $0 < x < \pi$. By inspection of Figure 5.1, we see that the main contribution to the integral in (5.8) comes from the "big eyetooth" near $\vartheta = 0$. In fact, the formal $n \to \infty$ limit of the integral (5.8) is

$$\int_0^\infty \frac{\sin u}{2u} du = \frac{\pi}{4}. \tag{5.9}$$

The numerical value $\frac{\pi}{4}$ is coughed up by contour integration. Hence, the formal $n \to \infty$ limit of (5.8) with x fixed in $0 < x < \pi$ is

$$\frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\} = 1. \tag{5.10}$$

Due to the 2π periodicity and odd symmetry of $\sin x, \sin 3x, \dots,$ we see that the infinite series on the left-hand side represents a "square wave" $S(x)$ on $-\infty < x < \infty$ whose graph is depicted in Figure 5.2. The zero values at

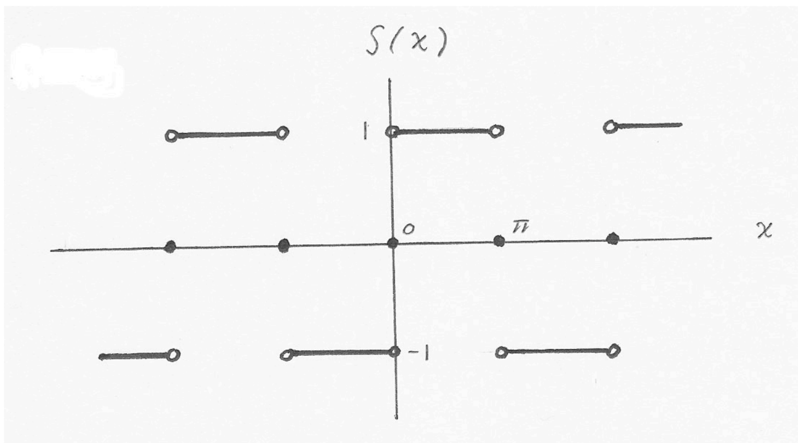

Figure 5.2

$x = n\pi$ follow from $\sin n\pi = \sin 3n\pi = \dots = 0$.

Let $T(x)$ be the antiderivative of $S(x)$ with $T(0) = 0$. The graph of $T(x)$ is the "triangle wave" depicted in Figure 5.3. Term-by-term integration of

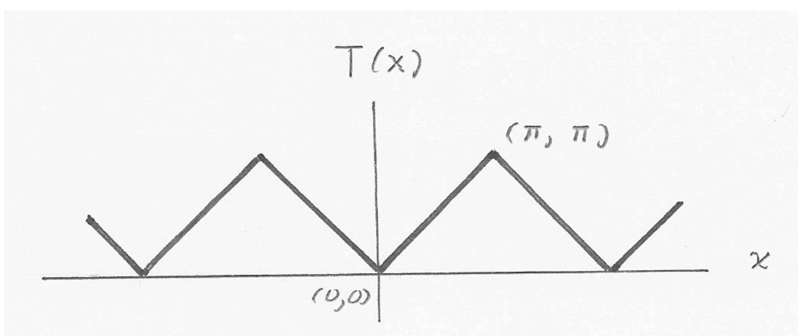

Figure 5.3

$$S(x) = \frac{4}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right\} \tag{5.11}$$

gives

$$T(x) = C - \frac{4}{\pi}\left\{\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right\},$$

where C is a constant of integration. We can evaluate C by examining the *average value* of $T(x)$: Think of the graph in Figure 5.3 as "triangular mounds of dirt". We see that "leveling the peaks and dumping the fill into the trenches" leads to a "level surface of elevation $\frac{1}{2}$". Hence, $C = \frac{1}{2}$, and we conclude that the triangle wave has Fourier series

$$T(x) = \frac{1}{2} - \frac{4}{\pi}\left\{\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right\}. \tag{5.12}$$

As an amusing curiosity, notice that setting $T(0) = 0$ in (5.11) leads to

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}. \tag{5.13}$$

If we know that $f(x)$ has a Fourier series as in (5.3), it is easy to determine the *Fourier coefficients* $\hat{f}(k)$. Let's pick off $\hat{f}(k)$ for a particular k: First, change the variable of summation in (5.3) to k'. Then multiply both sides of the equation by e^{-ikx}, and finally, integrate over $-\pi < x < \pi$. We find

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \sum_{k'=-\infty}^{\infty} \hat{f}(k') \int_{-\pi}^{\pi} e^{i(k'-k)x}dx. \tag{5.14}$$

In the right-hand side we formally exchanged the order of k' summation and x-integration. The integral on the right-hand side is elementary:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-k')x}dx = \delta_{kk'}, \tag{5.15}$$

where $\delta_{kk'}$ denotes the *Kronecker delta*

$$\delta_{kk'} := \begin{cases} 1, & k = k', \\ 0, & k \neq k'. \end{cases} \tag{5.16}$$

Hence, the right-hand side of (5.14) reduces to

$$\sum_{k'=-\infty}^{\infty} \hat{f}(k')2\pi\delta_{kk'} = 2\pi\hat{f}(k), \tag{5.17}$$

and we have

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx. \tag{5.18}$$

This little calculation of $\hat{f}(k)$ is the easy part. The deeper business is to spell out the class of $f(x)$ so that the Fourier series (5.3) with the coefficients (5.18) actually converges to $f(x)$. The inclusion of ever crazier $f(x)$ can lead to extreme elaboration and technicalities. Here, consider 2π periodic, piecewise continuous $f(x)$ with a finite number of jump discontinuities in a period interval of length 2π. Then the Fourier series (5.3) converges to $f(x)$ at x where $f(x)$ is continuous, and to the average of "left" and "right" values at a jump discontinuity $x = x_*$. The average in question is

$$\frac{f(x_*^-) + f(x_*^+)}{2}.$$

Our previous constructions of square and triangle waves $S(x)$ and $T(x)$ illustrate the general result.

Gibb's phenomenon

refers to the *non-uniform convergence* of the Fourier series as x approaches a jump discontinuity of $f(x)$. The Fourier series (5.11) of the square wave gives the clearest illustration: Consider the partial sum of (5.11),

$$S_n(x) := \frac{4}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \dots \frac{1}{2n-1} \sin(2n-1)x \right\}. \tag{5.19}$$

Previously, we gave a plausibility argument that $S_n(x) \to S(x) = 1$ as $n \to \infty$ with x in $0 < x < \pi$ fixed. Now consider a different limit process, with $n \to \infty$ and $x = \frac{\pi}{2n} \to 0$ as $n \to \infty$. From (5.8) we have

$$S_n\left(\frac{\pi}{2n}\right) = \int_0^{\pi} \frac{\sin \vartheta}{\pi n \sin \frac{\vartheta}{2n}} d\nu. \tag{5.20}$$

Figure 5.4 is a graph of the integrand. The shaded area is $S_n\left(\frac{\pi}{2n}\right)$. The area under the chord from $\left(0, \frac{2}{\pi}\right)$ to $(\pi, 0)$ is *one*. Hence, for all n, we have $S_n\left(\frac{\pi}{2n}\right) > 1$. In the limit $n \to \infty$,

$$S_n\left(\frac{\pi}{2n}\right) \simeq \frac{2}{\pi} \int_0^{\pi} \frac{\sin \vartheta}{\vartheta} d\vartheta \simeq 1.08.$$

Figure 5.5 shows what this means: No matter how large n is, there is a range of $x = O\left(\frac{1}{n}\right)$ where the partial sum $S_n(x)$ "overshoots" the exact square value value 1 by a finite amount, independent of x.

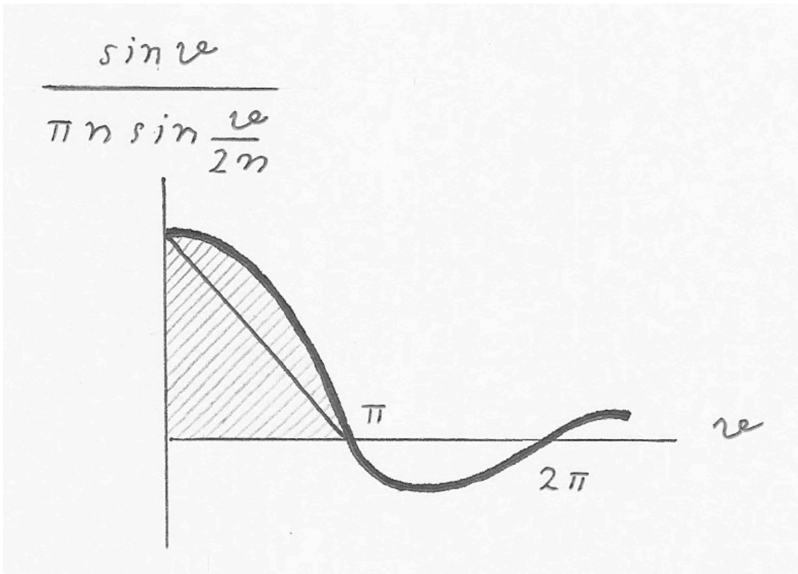

Figure 5.4

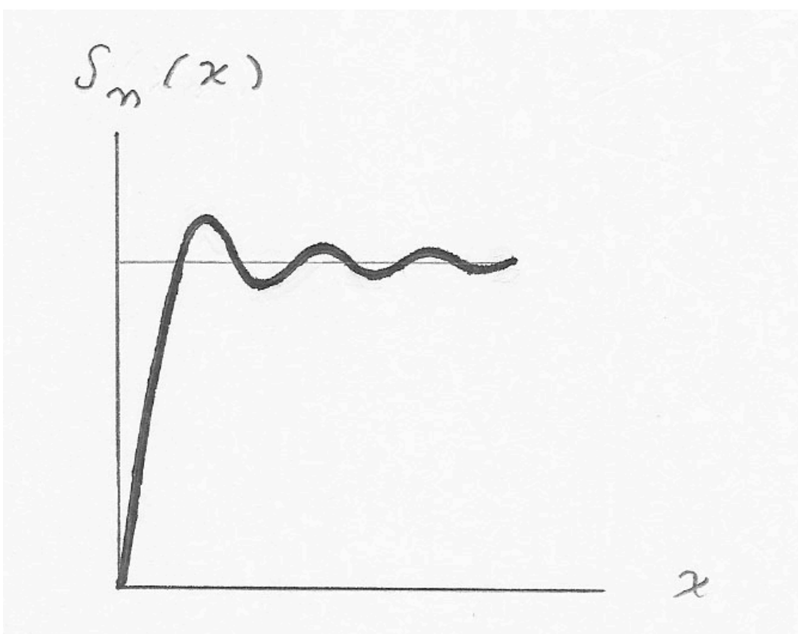

Figure 5.5

We present a quintessential application of Fourier series.

Output kernel

Figure 5.6 depicts a resistor and capacitor in series. The voltage at the

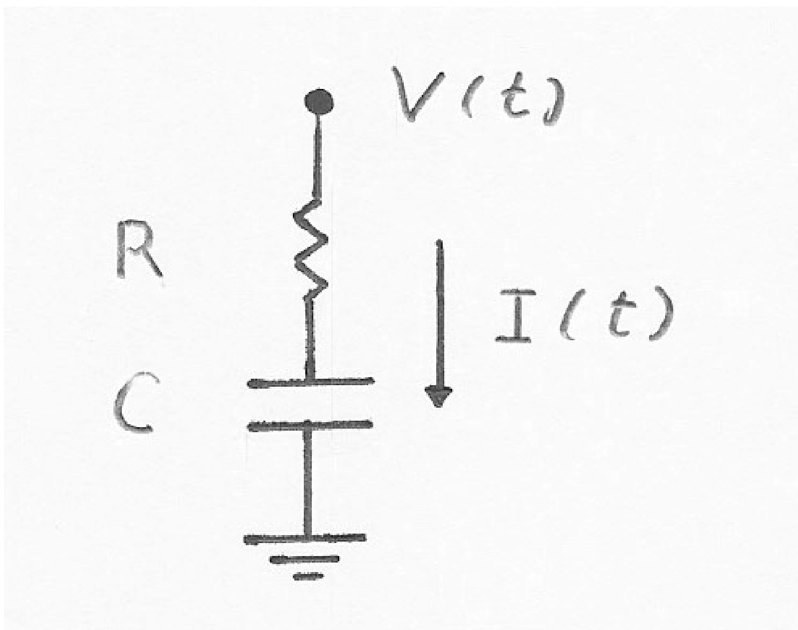

Figure 5.6

top node is periodic in time with angular frequency ω. The voltage is 2π periodic in the dimensionless time $\theta := \omega t$, and can be represented as the real part of the complex Fourier series

$$V(\theta) = \sum_{-\infty}^{\infty} \hat{V}_k e^{ik\theta}. \tag{5.21}$$

We want to determine the total current to ground, also 2π periodic in θ. The current is the real part of another Fourier series

$$I(\theta) = \sum_{-\infty}^{\infty} \hat{I}_k e^{ik\theta}. \tag{5.22}$$

For each k, the voltage and current amplitudes $\hat{V}_k$ and $\hat{I}_k$ are related by

$$\hat{V}_k = Z(k\omega)\hat{I}_k \tag{5.23}$$

where $Z(k\omega)$ is the impedance of the RC circuit at frequency $k\omega$,

$$Z(k\omega) = R + \frac{1}{ik\omega C} = R\left(1 + \frac{1}{i\Omega k}\right). \tag{5.24}$$

Here,

$$\Omega := RC\omega \tag{5.25}$$

is the frequency in units of $\frac{1}{RC}$. Solving (5.23) for $\hat{I}_k$, we have

$$\hat{I}_k = \frac{\hat{V}_k}{Z(k\omega)} = \frac{1}{R}\left(1 - \frac{1}{1 + i\Omega k}\right)\hat{V}_k. \tag{5.26}$$

The complex current (5.22) becomes

$$I = \sum_{-\infty}^{\infty} \frac{1}{R}\left(1 - \frac{1}{1 + i\Omega k}\right)\hat{V}_k e^{ik\theta}, \tag{5.27}$$

or recognizing

$$\sum_{-\infty}^{\infty} \frac{\hat{V}_k}{R} e^{ik\theta} = \frac{V(\theta)}{\Omega},$$

we have (5.28)

$$I = \frac{V(\theta)}{R} - \sum_{-\infty}^{\infty} \frac{\hat{V}_k}{1 + i\Omega k} e^{ik\theta}.$$

In principle $I = I(\theta)$ is given by the Fourier series (5.27), and taking the $\hat{V}_k$ as given, we can say "we're done".

Alternatively, we can find an integral operator which takes $V(\theta)$ as input, and produces $I(\theta)$ as output, with no reference to Fourier series at all. Of course, we are going to derive the integral operator *from* the Fourier series.

The derivation starts by inserting into (5.28) the formula for the voltage Fourier coefficients,

$$\hat{V}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(\theta')e^{-ik\theta'}d\theta'. \tag{5.29}$$

We obtain

$$I(\theta) = \frac{V(\theta)}{R} - \frac{1}{R}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{-\infty}^{\infty}\frac{e^{ik(\theta-\theta')}}{1+i\Omega k}\right)V(\theta')d\theta',$$

or introducing the *kernel*

$$F(\theta) := \frac{1}{2\pi}\sum_{-\infty}^{\infty}\frac{e^{ik\theta}}{1+i\Omega k}\,, \tag{5.30}$$

the current is given by

$$I(\theta) = \frac{V(\theta)}{R} - \frac{1}{R}\int_{-\pi}^{\pi}F(\theta-\theta')V(\theta')d\theta'. \tag{5.31}$$

The right-hand side is the aforementioned integral operator.

The kernel (5.30) is still in the form of a Fourier series. We can explicitly sum this Fourier series: First observe that the Fourier coefficients $\hat{F}_k$ of $F(\theta)$ are

$$\hat{F}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}F(\theta)e^{-ik\theta}d\theta = \frac{1}{2\pi}\frac{1}{1+i\Omega k}\,,$$

and it follows that for each k,

$$\int_{-\pi}^{\pi}F(\theta)e^{-ik\theta}d\theta = \frac{1}{1+i\Omega k}\,. \tag{5.32}$$

The determination of $F(\theta)$ from (5.32) is not so hard as it looks. Observe that the indefinite integral

$$\int\frac{1}{\Omega}e^{-\left(\frac{1}{\Omega}+ik\right)\theta}d\theta = -\frac{e^{-\left(\frac{1}{\Omega}+ik\right)\theta}}{1+i\Omega k} \tag{5.33}$$

produces the factor $\frac{1}{1+i\Omega k}$ in the right-hand side of (5.32). This motivates the proposal, of piecewise continuous $F(\theta)$ in the form

$$F(\theta) = \begin{cases} \dfrac{a_-}{\Omega}e^{-\frac{\theta}{\Omega}}, & -\infty < \theta < 0, \\ \dfrac{a_+}{\Omega}e^{-\frac{\theta}{\Omega}}, & 0 < \theta < \pi. \end{cases} \tag{5.34}$$

We find from (5.33), (5.34) that

$$\begin{aligned}\int_{-\pi}^{\pi} F(\theta)e^{-ik\theta}d\theta &= -\frac{a_-}{1+i\Omega k}\left[e^{-\left(\frac{1}{\Omega}+ik\right)\theta}\right]_{-\pi}^{0} - \frac{a_+}{1+i\Omega k}\left[e^{-\left(\frac{1}{\Omega}+ik\right)\theta}\right]_{0}^{\pi} \\ &= \frac{1}{1+i\Omega k}\left\{a_+ - a_- - e^{ik\pi}\left(a_+e^{-\frac{\pi}{\Omega}} - a_-e^{\frac{\pi}{\Omega}}\right)\right\}.\end{aligned} \tag{5.35}$$

(5.35) reduces to (5.32) if

$$a_+ - a_- = 1,\ a_+e^{-\frac{\pi}{\Omega}} - a_-e^{\frac{\pi}{\Omega}} = 0,$$

and from these, we deduce

$$a_- = -\frac{e^{-\frac{\pi}{\Omega}}}{2\sinh\frac{\pi}{\Omega}},\ a_+ = -\frac{e^{\frac{\pi}{\Omega}}}{2\sinh\frac{\pi}{\Omega}}.$$

In summary, the kernel $F(\theta)$ is

$$F(\theta) = \begin{cases} -\dfrac{e^{-\frac{\pi+\theta}{\Omega}}}{2\sinh\frac{\pi}{\Omega}}, & -\pi < \theta < 0, \\ -\dfrac{e^{\frac{\pi-\theta}{\Omega}}}{2\sinh\frac{\pi}{\Omega}}, & 0 < \theta < \pi. \end{cases} \tag{5.36}$$

Figure 5.7 is the graph of $F(\theta)$. Notice that F and its derivative F' have the same values at $\theta = +\pi$ and $-\pi$, so the periodic extension of $F(\theta)$ in (5.36) to all θ has jump discontinuities only at $\theta = 2\pi n$, $n =$ integer. The simple piecewise exponential form of $F(\theta)$ suggests some simple, direct derivation, independent of Fourier analysis. We'll do this in chapter 6.

Fourier transform

We go right to the main point: If $f(x)$ on $-\infty < x < \infty$ has a *Fourier transform* $\hat{f}(k)$ on $-\infty < k < \infty$ so that $f(x)$ has the *Fourier integral representation* (5.6), then what is $\hat{f}(k)$? We are tempted to imitate the calculation (5.14)–(5.18) of Fourier coefficients $\hat{f}(k)$. After all, the difference between Fourier series and integral representations of $f(x)$ in (5.3), (5.6) is the replacement of summation over $k =$ integers in (5.3) by k-integration in (5.6). The imitation starts like this: We want to pick off $\hat{f}(k)$ for some k:

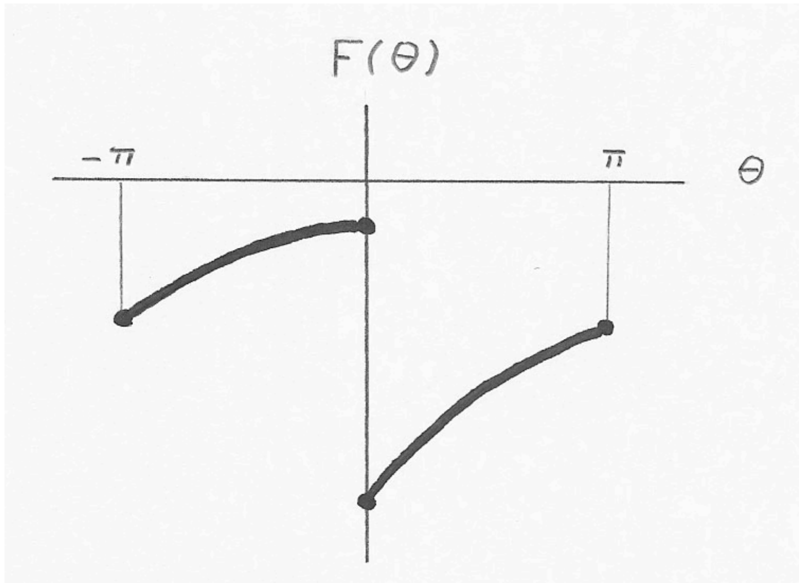

Figure 5.7

Replace the variable of integration k in (5.6) by k', multiply the resulting equation by e^{-ikx}, and x-integrate over $-\infty < x < \infty$. We obtain

$$\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \int_{-\infty}^{\infty} \hat{f}(k') \left(\int_{0}^{\infty} e^{i(k'-k)x}dx \right) dk'. \qquad (5.37)$$

In the right-hand side we interchanged the orders of k' and x integrations. The x-integral in the right-hand side is presumably a function of $k'-k$. That is,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x}dx = \delta(k'-k), \qquad (5.38)$$

where $\delta(k)$ is the *Dirac delta function* defined formally by

$$\delta(k) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}dx. \qquad (5.39)$$

$\delta(k'-k)$ is the formal analog of the Kronecker delta in (5.15)–(5.16). Combining (5.21), (5.22), we have

$$\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = 2\pi \int_{-\infty}^{\infty} \hat{f}(k')\delta(k'-k)dk'. \qquad (5.40)$$

In (5.17), we see how the Kronecker delta "picks out the one term $2\pi\hat{f}(k)$ from the sum in the left-hand side". Presumably, the Dirac delta function does the analogous job to the k' integral in the right-hand side of (5.40),

$$\int_{-\infty}^{\infty} \hat{f}(k')\delta(k'-k)dk' = \hat{f}(k). \tag{5.41}$$

If you believe all of this, you obtain from (5.24), (5.25) the *inversion formula*,

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \tag{5.42}$$

(5.42) is the analog of the formula (5.18) for coefficients of the Fourier series.

The essence of this formal analysis leading to (5.42) is the "selection identity" (5.41) which presumably holds for some large class of $\hat{f}(k)$. Now how can this be? The usual explanation is that $\delta(k'-k)$ vanishes for all $k' \neq k$, but nevertheless the area under the graph of $\delta(k'-k)$ versus k is *one*, all concentrated at the single point $k'-k$. That is, the Dirac delta function in (5.23) is supposed to vanish for all $k \neq 0$,

$$\delta(k) \equiv 0, \;\; k \neq 0, \tag{5.43}$$

but nevertheless,

$$\int_a^b \delta(k)dx = 1, \tag{5.44}$$

for any interval $[a,b]$ containing $x=0$. If you swallow (5.43) and (5.44), the argument for (5.41) goes like this: "Replacing $\hat{f}(k')$ in the left-hand side by $\hat{f}(k)$ makes no difference, since the integrand *vanishes* for all $k' \neq k$. We then remove the constant $\hat{f}(k)$ outside of the integral, and presto bingo we are left with

$$\hat{f}(k)\int_{-\infty}^{\infty} \delta(k'-k)dk' = \hat{f}(k),$$

by (5.44). You might recognize the preceding discussion as the usual "physics course explanation" of the Dirac delta function. If not, you have just received the whole nine yards.

This "derivation" of the inversion formula (5.42) has a well-known critique: The integral (5.39) which "defines" the "Dirac delta function" $\delta(k)$ is divergent. Furthermore, given that $\delta(k) \equiv 0$ for all $x \neq 0$, the assignment of *any* finite value for $\delta(0)$ gives $\int_a^b \delta(k)dx = 0$ for any a, b, $a < 0 < b$, and not $\int_a^b \delta(k)dx = 1$ as in (5.44). In this case, the "selection identity" doesn't happen. Nevertheless, in the apparently misguided analysis we "see through a glass darkly" first hints of what the delta function really is, and its role in a real derivation of the inversion formula.

A modified derivation called *Gaussian summation* starts out by multiplying the equation

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(k')e^{ik'x}dk'$$

by

$$e^{-ikx-\varepsilon^2 x^2}$$

with $\varepsilon > 0$, integrating both sides over $-\infty < x < \infty$, and exchanging orders of k' and x integrations in the right-hand side. In place of (5.37), we have

$$\int_{-\infty}^{\infty} f(x)e^{-ikx-\varepsilon^2 x^2}dx = 2\pi \int_{-\infty}^{\infty} \hat{f}(k')\delta(k-k',\varepsilon)dk', \tag{5.45}$$

where

$$\delta(k,\varepsilon) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\varepsilon^2 x^2}dx. \tag{5.46}$$

Unlike (5.39), the integral (5.46) is a well-defined function of k for any $\varepsilon > 0$. In fact, we can evaluate (5.46) by completing the square, as we did for the "Gaussian wavepacket" in chapter 4. We have

$$-\varepsilon x^2 - ikx = -\varepsilon\left(x - \frac{ik}{2\varepsilon^2}\right)^2 - \frac{k^2}{4\varepsilon^2},$$

so

$$\delta(k,\varepsilon) = \frac{1}{2\pi}e^{-\frac{k^2}{4\varepsilon^2}}\int_{-\infty}^{\infty} e^{-\varepsilon^2\left(x-\frac{ik}{2\varepsilon^2}\right)^2}dx = \frac{e^{-\frac{k^2}{4\varepsilon^2}}}{2\sqrt{\pi}\varepsilon}. \tag{5.47}$$

Figure 5.8 is the graph of $\delta(k,\varepsilon)$. Its a "spike" of height $\delta(0,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}$ and width $\Delta k = 4\varepsilon$. Notice that

$$\lim_{\varepsilon\to 0}\delta(k,\varepsilon) = 0,\ \ k \neq 0 \text{ fixed}, \tag{5.48}$$

but for any $\varepsilon > 0$, the area under the graph is *one*,

$$\int_{-\infty}^{\infty} \delta(k,\varepsilon)dk = 1. \tag{5.49}$$

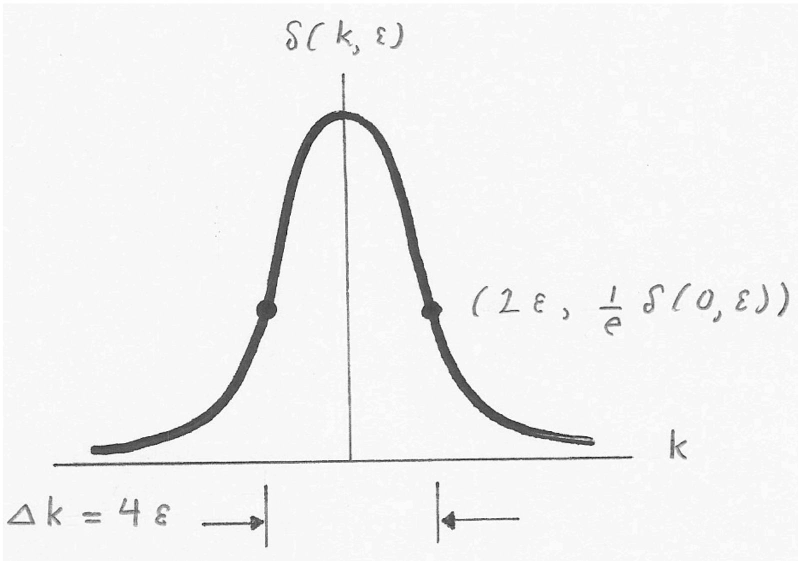

Figure 5.8

We see that the traditionally stated properties (5.43), (5.44) of the Dirac delta function really refer to an ε-*sequence* of functions whose graphs become "narrower and taller spikes" as $\varepsilon \to 0$ in such a way that $\delta(k, \varepsilon) \to 0$ as $\varepsilon \to 0$ with $k \neq$ fixed as in (5.48), but the area under the spike for any $\varepsilon > 0$ remains *one*, as in (5.49). An ε-sequence of functions with these characteristics is called a *delta sequence*. There are many other delta sequences which routinely arise in scientific calculations. Some have already been seen in the chapter 4 discussion on wavepackets. We'll get back to them. Here, we return to (5.45) with $\delta(k, \varepsilon)$ as in (5.41) and clean up the derivation of the inversion formula (5.42). A main point is how the delta sequence $\delta(k, \varepsilon)$ in place of $\delta(k)$ illuminates the "selection identity" (5.41): We have

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{f}(k')\delta(k-k', \varepsilon)dk' &= \int_{-\infty}^{\infty} \hat{f}(k')\frac{e^{-\frac{(k-k')^2}{4\varepsilon^2}}}{2\sqrt{\pi}\varepsilon}dk' \\ &= \int_{-\infty}^{\infty} \hat{f}(k+\varepsilon K)\frac{1}{2\sqrt{\pi}}e^{-\frac{K^2}{4}}dK.\end{aligned} \tag{5.50}$$

The last equality uses the change of integration variable $K := \frac{k'-k}{\varepsilon}$. In the limit $\varepsilon \to 0$, the right-hand side converges to

$$\hat{f}(k)\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{K^2}{4}} dK = \hat{f}(k).$$

In summary,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \hat{f}(k')\delta(k-k',\varepsilon)dk' = \hat{f}(k). \tag{5.51}$$

(5.51) is the rigorous replacement of the "selection identity" (5.41). By virtue of (5.51), (5.45) reduces to the inversion formula (5.42) as $\varepsilon \to 0$.

Superposition of point source solutions

$u(x,t)$ satisfies the damped wave equation with a time periodic source,

$$u_{tt} + \beta u_t - u_{xx} = f(x)e^{i\omega t}. \tag{5.52}$$

Here, β is a positive damping coefficient, and the function $f(x)$ is localized about $x = 0$. That is, $f(x) = 0$ for $|x|$ sufficiently large. Physical wavefields are obtained as the real part of complex solutions for $u(x,t)$. There are time periodic solutions of the form

$$u(x,t) = U(x)e^{i\omega t}, \tag{5.53}$$

where $U(x)$ satisfies the ODE

$$U'' + (\omega^2 - i\beta\omega)U = -f(x). \tag{5.54}$$

Due to the damping, we expect that physically relevant solutions have $U \to 0$ as $|x| \to \infty$. We seek a Fourier integral representation of the solution

$$U(x) = \int_{-\infty}^{\infty} \hat{U}(k)e^{ikx}dk. \tag{5.55}$$

Substituting (5.55) into (5.54), we find

$$\int_{-\infty}^{\infty} \{(-k^2 + \omega^2 - i\beta\omega)\hat{U}(k) - \hat{f}(k)\}e^{ikx}dk = 0. \tag{5.56}$$

Here, $\hat{f}(k)$ is the Fourier transform of $f(x)$. We satisfy (5.56) by taking

$$\hat{U}(k) = \frac{\hat{f}(k)}{\omega^2 - i\beta\omega - k^2}\,. \tag{5.57}$$

Back substitution of $\hat{U}(k)$ in (5.57) into (5.55) gives

$$U(x) = \int_{-\infty}^{\infty} \frac{\hat{f}(k)}{\omega^2 - i\beta\omega - k^2} e^{ikx} dk. \tag{5.58}$$

If $f(x)$ is given, then in principle its Fourier transform $\hat{f}(k)$ is determined and (5.58) is our "answer" for $U(x)$. (5.58) is analogous to the Fourier series (5.27) for the current, which is "determined" by the Fourier coefficients $\hat{V}_k$ of the voltage.

Alternatively, we can determine $U(x)$ as an integral operator applied to $f(x)$. In (5.58) we substitute for $\hat{f}(k)$ the inversion formula

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx.$$

Interchanging the order of x' and k integrations, we obtain

$$U(x) = \int_{-\infty}^{\infty} f(x') \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{\omega^2 - i\beta\omega - k^2} dk \right) dx'. \tag{5.59}$$

Define

$$g(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{\omega^2 - i\beta\omega - k^2} dk. \tag{5.60}$$

The $U(x)$ in (5.59) becomes the *convolution* $(f * g)(x)$, defined by

$$U(x) = (f * g)(x) := \int_{-\infty}^{\infty} f(x') g(x - x') dx'. \tag{5.61}$$

Point source solution

The function $g(x)$ in (5.60), (5.61) has a special significance. In the convolution formula (5.61), take $f(x)$ to be a delta sequence $\delta(x - X, \varepsilon)$, centered about $x = X$, as depicted in Figure 5.9. As ε approaches zero, the source in (5.52) concentrates about the point $x = X$, but retains "unit strength" by virtue of $\int_{-\infty}^{\infty} \delta(x - X, \varepsilon) dx = 1$ for $\varepsilon > 0$. In this sense, we have "a point source of unit strength" in the limit $\varepsilon \to 0$. The $\varepsilon \to 0$ limit of $U(x)$ in (5.61) is

$$U(x) \to \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \delta(x' - X, \varepsilon) g(X - x') dx' = g(x - X). \tag{5.62}$$

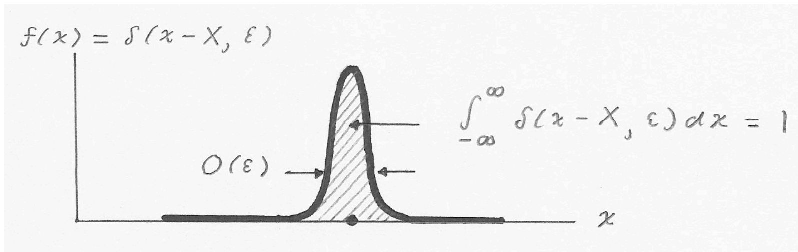

Figure 5.9

We think of $g(x-X)$ as "the solution for $U(x)$ due to a point source of unit strength at $x = X$". The convolution (5.61) can be regarded as a *superposition of point source solutions.*

We evaluate $g(x)$ in (5.60). We see that the Fourier transform of $g(x)$ is $\hat{g}(k) = \frac{1}{2\pi}\frac{1}{\omega^2-\beta i\omega-k^2}$, so inversion of (5.60) gives

$$\frac{1}{\omega^2 - i\beta\omega - k^2} = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx. \tag{5.63}$$

Analogous to the Fourier series example, we seek $g(x)$ as a piecewise exponential. Since $\hat{g}(k)$ is even in k, $g(x)$ is even in x, so the piecewise exponential form of $g(x)$ is

$$g(x) = Ce^{-z|x|} \tag{5.64}$$

for some complex constants C and z. We have $\operatorname{Re} z > 0$, so $g(x)$ vanishes as $|x| \to\infty$. Given $g(x)$ in (5.64), we calculate

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)e^{-ikx}dx &= C\int_0^{\infty} e^{-(z+ik)x}dx + C\int_{-\infty}^{0} e^{(z-ik)x}dx \\ &= \frac{C}{z+ik} + \frac{C}{z-ik} = \frac{2Cz}{z^2+k^2}\,.\end{aligned} \tag{5.65}$$

Comparing (5.63), (5.65), we see that

$$z^2 = -\omega^2 + i\beta\omega,\ C = -\frac{1}{2z}\,. \tag{5.66}$$

The choice for z with positive real part is

$$z = i\omega\sqrt{1-\frac{i\beta}{\omega}}, \tag{5.67}$$

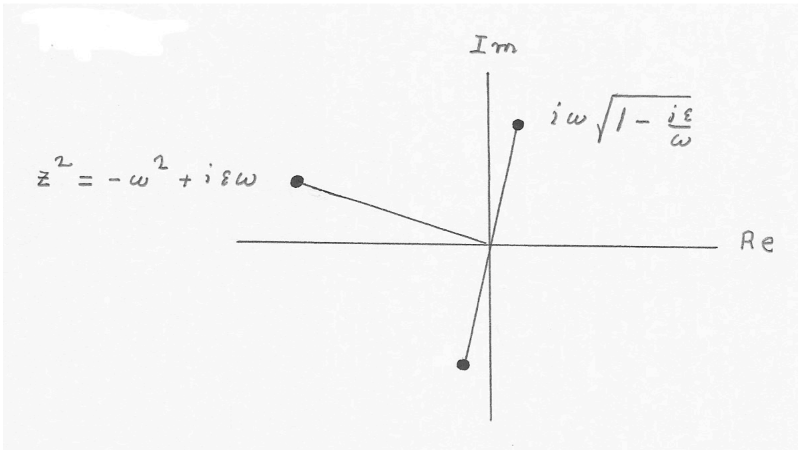

Figure 5.10

where $\sqrt{1-\frac{i\beta}{\omega}}$ is unity for $\varepsilon = 0$. Figure 5.10 depicts the construction of z in (5.67). With z and C as in (5.66), (5.67), the final result for $g(x)$ is

$$g(x) = \frac{1}{2i\omega\sqrt{1-\frac{i\beta}{\omega}}} e^{-i\omega\sqrt{1-\frac{i\beta}{\omega}}|x|}. \tag{5.68}$$

The corresponding wavefield $u(x,t)$ is

$$u(x,t) = \frac{1}{2i\omega\sqrt{1-\frac{i\beta}{\omega}}} e^{i\omega\left(t-\sqrt{1-\frac{i\beta}{\omega}}|x|\right)}. \tag{5.69}$$

Notice that $u(x,t)$ is a damped *right* traveling wave in $x > 0$, and a damped *left* traveling wave in $x < 0$. Since the waves are radiating *away* from the source at $x = 0$, we say the waves are *outgoing*. Figure 5.11 depicts a snapshot of $\operatorname{Re} u(x,t)$ versus x for fixed t.

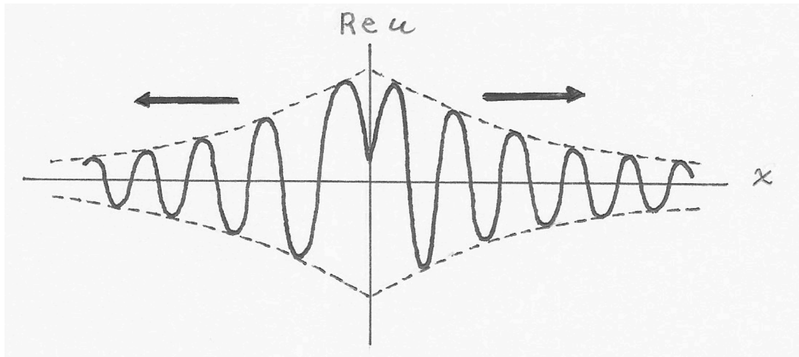

Figure 5.11