Chapter 4

The complex exponential in science

Superposition of oscillations and beats

In a meditation hall, there was a beautiful, perfectly circular brass bowl. When you struck it with the leather covered hammer, it produced a beautiful pure tone. The pressure variation δp of the sound was a pure sinusoid of some angular frequency ω:

$$\delta p \;\; \alpha \quad \cos \omega t = \operatorname{Re} e^{i\omega t}. \tag{4.1}$$

Given any standing wave of the bowl, you could get another of exactly the pitch by rotation about the axis of circular symmetry. But then came a barbarian which whacked the bowl really hard and he put a dent in it. No more perfect circular symmetry. Now there are two distinct standing waves, with two close but nevertheless different frequencies ω_1 and ω_2 with $|\omega_1 - \omega_2| \ll \frac{\omega_1+\omega_2}{2}$. When you hit the dented bowl, you get a *superposition* (that is, linear combination) of the two standing waves, and the sound you hear is a superposition of two sinusoids with frequencies ω_1 and ω_2. In place of (4.1) we have

$$\delta p = \operatorname{Re}\{a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}\} \tag{4.2}$$

where a_1 and a_2 are complex constants. Due to the difference between the frequencies ω_1 and ω_2, the two vibrations in (4.2) slowly drift from "in phase" when they add or "constructively interfere" to "out of phase", when they cancel each other, or "destructively interfere". This is the familiar phenomenon of *beats*. Let's look at (4.2) with $a_1 = a_2 = 1$. Introducing the phases

$\theta_1 := \omega_1 t$ and $\theta_2 := \omega_2 t$, (4.2) with $a_1 = a_2 = 1$ reads

$$\delta p = \mathrm{Re}\{e^{i\theta_1} + e^{i\theta_2}\}. \tag{4.3}$$

To see the beats "hiding in (4.3)", we "factor out the average of the phases". That is, rewrite (4.3) as

$$\begin{aligned}\delta p &= \mathrm{Re}\left\{ e^{i\frac{\theta_1+\theta_2}{2}} \left(e^{i\frac{\theta_1-\theta_2}{2}} + e^{-i\frac{\theta_1-\theta_2}{2}} \right) \right\} \\ &= \mathrm{Re}\left\{ e^{i\frac{\theta_1+\theta_2}{2}} 2\cos\left(\frac{\theta_1-\theta_2}{2}\right) \right\} \\ &= 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right),\end{aligned}$$

or finally,

$$\delta p = 2\cos\left(\frac{\omega_1+\omega_2}{2}\,t\right)\cos\left(\frac{\omega_1+\omega_2}{2}\,t\right). \tag{4.4}$$

Figure 4.1 is the graph of δp versus time based on (4.4). The beat period

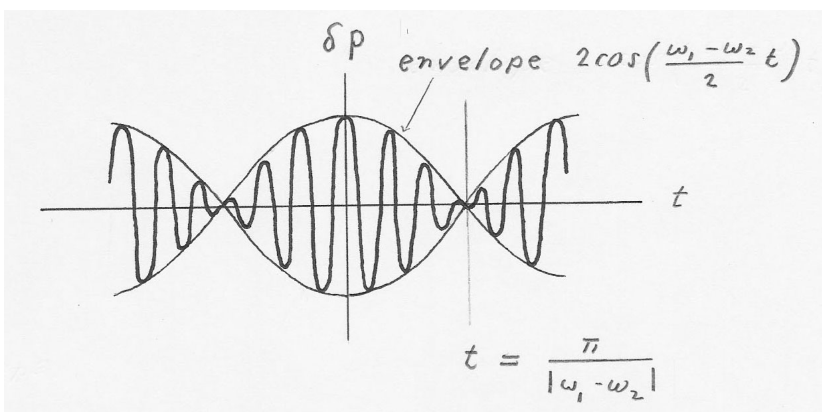

Figure 4.1

(time interval between moments of complete destructive interference) is

$$T = \frac{2\pi}{|\omega_1 + \omega_2|}.$$

If $|a_1| \neq |a_2|$ the analysis has to dig a little deeper. For instance, take $a_1 = 1$, $a_2 = \frac{1}{2}$. Then we are dealing with

$$\begin{aligned} e^{i\theta_1} + \frac{1}{2}\, e^{i\theta_2} &= e^{i\frac{\theta_1+\theta_2}{2}} \left(e^{i\frac{\theta_1-\theta_2}{2}} + \frac{1}{2} e^{-i\frac{\theta_1-\theta_2}{2}} \right) \\ &= e^{i\frac{\theta_1+\theta_2}{2}} \left(\frac{3}{2} \cos \frac{\theta_1 - \theta_2}{2} + \frac{i}{2} \sin \frac{\theta_1 - \theta_2}{2} \right). \end{aligned} \tag{4.5}$$

Next we express the "complex beat amplitude" in parentheses in polar form. That is, seek modulus R and argument ψ so

$$\mathrm{R}e^{i\psi} = \frac{3}{2} \cos \frac{\theta_1 - \theta_2}{2} + \frac{i}{2} \sin \frac{\theta_1 - \theta_2}{2}\,. \tag{4.6}$$

Then (4.5) becomes

$$e^{i\theta_1} + \frac{1}{2} e^{i\theta_2} = \mathrm{R}e^{i\left(\frac{\theta_1+\theta_2}{2} + \psi\right)}$$

and the real part is

$$R \cos \left(\frac{\theta_1 + \theta_2}{2} + \psi \right).$$

It is now clear that R as a function of $\theta_1 - \theta_2$ is the beat amplitude. From (4.6) we compute

$$R = \sqrt{\frac{9}{4} \cos^2 \left(\frac{\theta_1 - \theta_2}{2} \right) + \frac{1}{4} \sin^2 \left(\frac{\theta_1 - \theta_2}{2} \right)} = \sqrt{\frac{1}{4} + 2 \cos^2 \left(\frac{\theta_1 - \theta_2}{2} \right)},$$

or expressed as a function of time,

$$R = \sqrt{\frac{1}{4} + 2 \cos^2 \left(\frac{\omega_1 - \omega_2}{2}\, t \right)}.$$

Figure 4.2 shows the graph of δp versus time in (4.2) with $a_1 = 1$, $a_2 = \frac{1}{2}$.

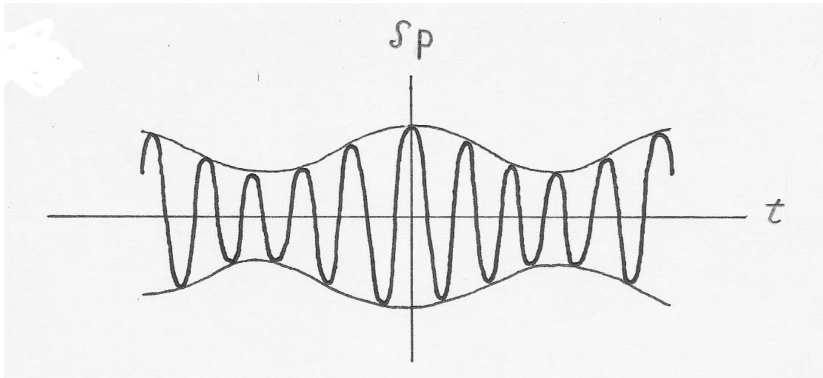

Figure 4.2

Waves, wave packets and group velocity

Consider the sinusoidal *traveling wave* in one space dimension

$$\psi = \cos(kx - \omega t) = \mathrm{Re}\{e^{i(kx-\omega t)}\}. \tag{4.7}$$

Here, k and ω are given constants. $\theta := kx - \omega t$ is called the *phase* of the wave. The level curves in (x, t) spacetime with $\theta \equiv 2\pi n$, $n =$ integer are "crests" where $\psi = +1$, $\theta \equiv 2\pi n + \pi$ correspond to "troughs" where $\psi = -1$. In Figure 4.3 we've plotted world lines of crests. If you sit at a fixed position x, crests pass by you, one per time period $\frac{2\pi}{\omega}$. The constant ω which measures the time rate of change of phase at fixed x is called the *angular frequency* of the wave. If you take a "snapshot" of the wave at fixed time, you'll observe the spatial period $\frac{2\pi}{k}$. The number of spatial periods in an interval of length L is $\frac{kL}{2\pi}$, so $\frac{k}{2\pi}$ is the "density of waves" seen at fixed time. Perhaps this is the reason for calling k the *wavenumber.* We see from (4.7) that a world line $x = x(t)$ of constant phase has

$$\dot{x} = v_p := \frac{\omega}{k} \tag{4.8}$$

v_p in (4.8) is called the *phase velocity.*

If we add two traveling waves whose wavenumbers k_1 and k_2 and frequencies ω_1 and ω_2 close to each other, we'll get "beats in spacetime": The sum

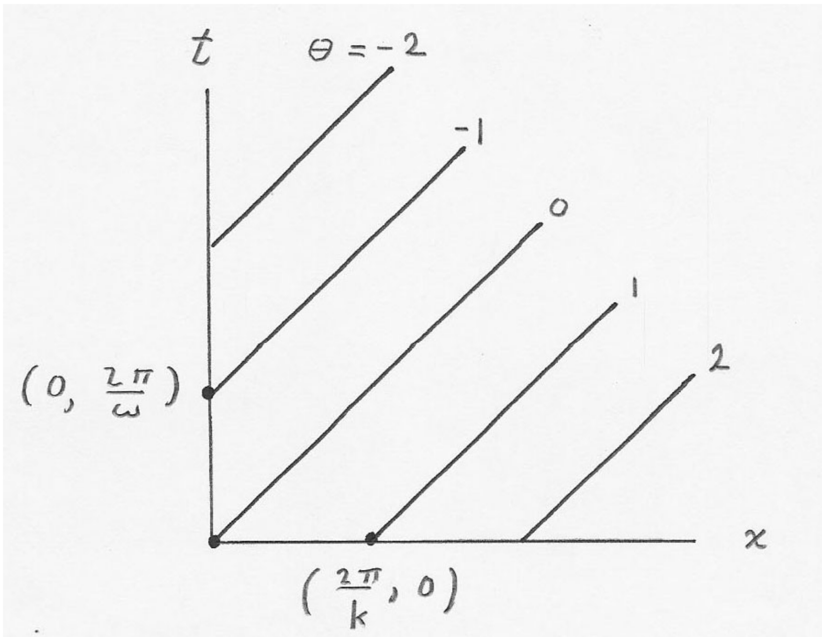

Figure 4.3

of two waves with the phases $\theta_1 = k_1 x - \omega_1 t$ and $\theta_2 = k_2 x - \omega_2 t$ is

$$\begin{aligned} \psi &= \mathrm{Re}\{e^{i\theta_1} + e^{i\theta_2}\} \\ &= \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \cos\left(\frac{\theta_1 - \theta_2}{2}\right) \\ &= \cos\left(\frac{k_1 + k_2}{2}\, x - \frac{\omega_1 + \omega_2}{2}\, t\right) \cos\left(\frac{k_1 - k_2}{2}\, x - \frac{\omega_1 - \omega_2}{2}\, t\right). \end{aligned} \tag{4.9}$$

The first factor

$$\cos\left(\frac{k_1 + k_2}{2}\, x - \frac{\omega_1 + \omega_2}{2}\, t\right) \tag{4.10}$$

has the form of a traveling wave whose phase velocity is $\frac{\omega_1+\omega_2}{k_1+k_2}$. (4.10) is often called "the carrier wave". In the limit ω_1, ω_2 both approaching ω, and k_1, k_2 approaching k, the carrier wave phase velocity converges to v_p in (4.8). The second factor in (4.9),

$$\cos\left(\frac{k_1 - k_2}{2}\, x - \frac{\omega_1 - \omega_2}{2}\, t\right) \tag{4.11}$$

is called the *envelope*. If $k_1 \neq k_2$, (4.11) represents a traveling wave of velocity $\frac{\omega_1-\omega_2}{k_1-k_2}$.

The physical context of the waves (4.7) often specifies a *dispersion relation*, so the frequency ω is some definite function of k, $\omega = \omega(k)$. Take $k_1 = k$, $\omega_1 = \omega(k)$, $k_2 = k+\kappa$ and $\omega_2 = \omega(k+\kappa)$. The envelope velocity is

$$\frac{\omega_2-\omega_1}{k_2-k_1} = \frac{\omega(k+\kappa)-\omega(k)}{\kappa},$$

which converges to the *group velocity*

$$v_g := \omega'(k) \tag{4.12}$$

as $\kappa \to 0$.

Here is a physical example, of water waves whose height is much less than the wavelength, and the wavelength is much less than the depth of ocean. The dispersion relation is

$$\omega(k) = \sqrt{gk}. \tag{4.13}$$

Here, g is the gravity acceleration, and $\sqrt{gk}$ is the only combination of g and k that has the physical unit of $1 \div$ time. From (4.13) we compute phase and group velocities,

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}}, \; v_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\, v_p.$$

Suppose you are in a fishing boat, and the wave pattern depicted in Figure 4.4 is approaching you. If you fix your attention on individual crests, you see

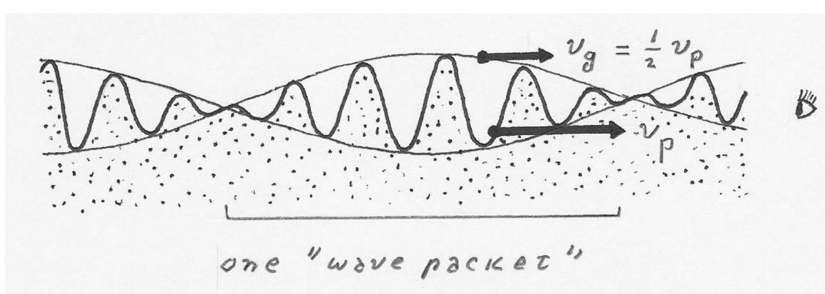

Figure 4.4

them emerge from the "rear end of a wave packet", as if from nothing, and disappear into the front end. This is because the crests moving at the phase velocity v_p are twice as fast as the envelope, which moves at the group velocity $v_g = \frac{1}{2}\,v_p$.

The superposition of just two traveling waves like (4.7) is *extremely special.* Much more commonly, we observe a superposition of waves with a continuous range of wavenumbers. Such a superposition is expressed as an integral,

$$\psi(x,t) = \mathrm{Re}\int_{-\infty}^{\infty} \hat{\psi}(k)e^{i(kx-\omega(k)t)}dk. \tag{4.14}$$

Think of (4.14) as a linear combination of individual traveling waves $e^{i(kx-\omega(k)t)}$ with k ranging over all real values, and $\hat{\psi}(k)$ represents the "coefficients" of the linear combination. We'll call $\hat{\psi}(k)$ the *spectrum* of the wavefield (4.14).

Let's take the spectrum confined to some narrow range of wavenumbers about some fixed K. For instance,

$$\hat{\psi}(k) = \begin{cases} \dfrac{1}{2\varepsilon} & \text{in } |k-K| < \varepsilon, \\[2ex] 0 & \text{in } |k-K| > \varepsilon. \end{cases} \tag{4.15}$$

Figure 4.5 is the graph of the spectrum $\hat{\psi}(k)$. The area under the graph is one,

$$\int_{-\infty}^{\infty} \hat{\psi}(k)dk = 1,$$

and its width is

$$\Delta k = 2\varepsilon. \tag{4.16}$$

First, let's look at the wavefield (4.14) at time zero,

$$\psi(x,0) = \mathrm{Re}\,\frac{1}{2\varepsilon}\int_{K-\varepsilon}^{K+\varepsilon} e^{ikx}dk. \tag{4.17}$$

By the integral formula (3.33) for the complex exponential, we have

$$\begin{aligned} \frac{1}{2\varepsilon}\int_{K-\varepsilon}^{K+\varepsilon} e^{ikx}dk &= \frac{1}{2\varepsilon ix}\{e^{i(K+\varepsilon)x} - e^{i(K-\varepsilon)x}\} \\ &= e^{iKx}\frac{\sin\varepsilon x}{\varepsilon x}\,, \end{aligned}$$

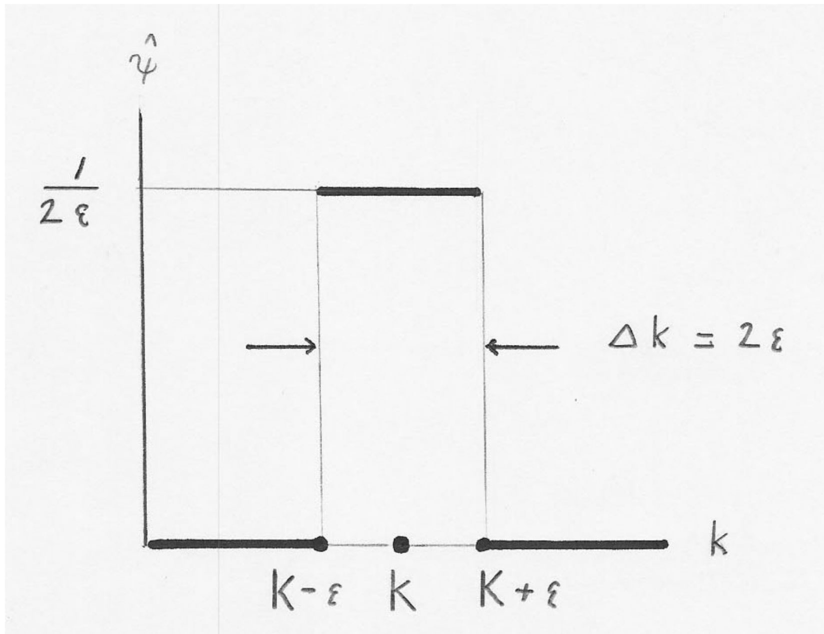

Figure 4.5

and (4.17) becomes

$$\psi(x,0) = \cos Kx \frac{\sin \varepsilon x}{\varepsilon x}\,. \tag{4.18}$$

Here, $\cos Kx$ is the "carrier wave" seen at $t = 0$, and $\frac{\sin \varepsilon x}{\varepsilon x}$ the envelope. An essential difference from the superposition of just two waves in (4.9) is that the envelope in (4.18) decays to zero as $|x| \to \infty$. The wavefield (4.14) is truly localized in space. Figure 4.6 depicts $\psi(x,0)$ in (4.12). The zeros of the envelope $\frac{\sin \varepsilon x}{\varepsilon x}$ closest to $x = 0$ define a characteristic width Δx

$$\Delta x = \frac{2\pi}{\varepsilon}\,. \tag{4.19}$$

Notice that the product of Δx and Δk in (4.16), (4.19) is independent of ε,

$$\Delta x \Delta k = 4\pi.$$

Qualitatively, we say that the widths of spectrum and wave packet are reciprocals to each other. This is generally true for superpositions of complex exponentials e^{ikx}.

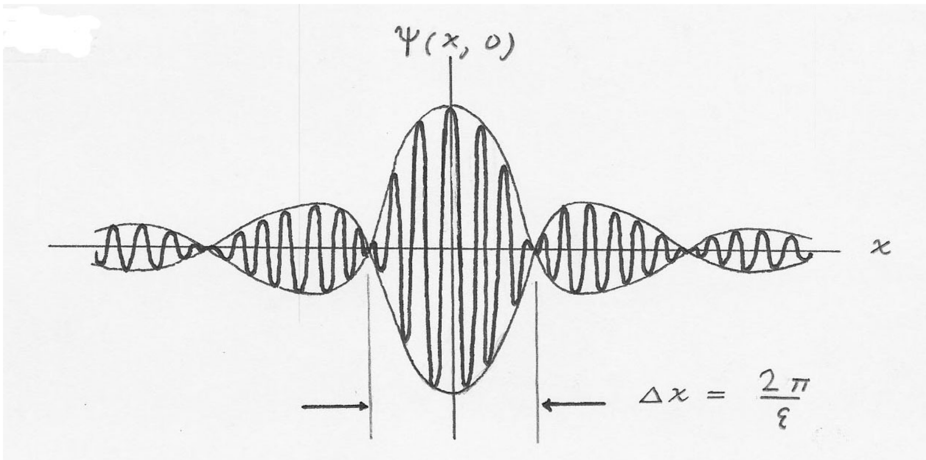

Figure 4.6

Here is another important example: The spectrum is the *Gaussian*,

$$\hat{\psi}(k) = \frac{1}{2\sqrt{\pi}\varepsilon} e^{-\frac{(k-K)^2}{4\varepsilon^2}}. \tag{4.20}$$

Figure 4.7 is is graph. The Gaussian (4.20) has certain similarities to the "skyscraper" shaped spectrum in (4.15). The area under the graph in Figure 4.7 is $\int_{-\infty}^{\infty} \hat{\psi}(k)dk = 1$, and the "width" is proportional to ε: In Figure 4.7, the labeled width $\Delta k = 2\varepsilon$ corresponds to the interval where $\hat{\psi}(k) > \frac{1}{e} \max \hat{\psi}$. Given the Gaussian spectrum (4.20), the wavefield (4.14) at $t = 0$ is

$$\psi(x, 0) = \text{Re}\, \frac{1}{2\sqrt{\pi}\varepsilon} \int_{-\infty}^{\infty} e^{-\frac{(k-K)^2}{4\varepsilon^2} + ikx} dk.$$

Changing the variable of integration to

$$u := \frac{k - K}{2\varepsilon} \tag{4.21}$$

gives

$$\begin{aligned} \psi(x, 0) &= \text{Re}\, \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2 + i(K+2\varepsilon u)x} du \\ &= \text{Re}\, \frac{e^{iKx}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2 + 2u(i\varepsilon x)} du. \end{aligned} \tag{4.22}$$

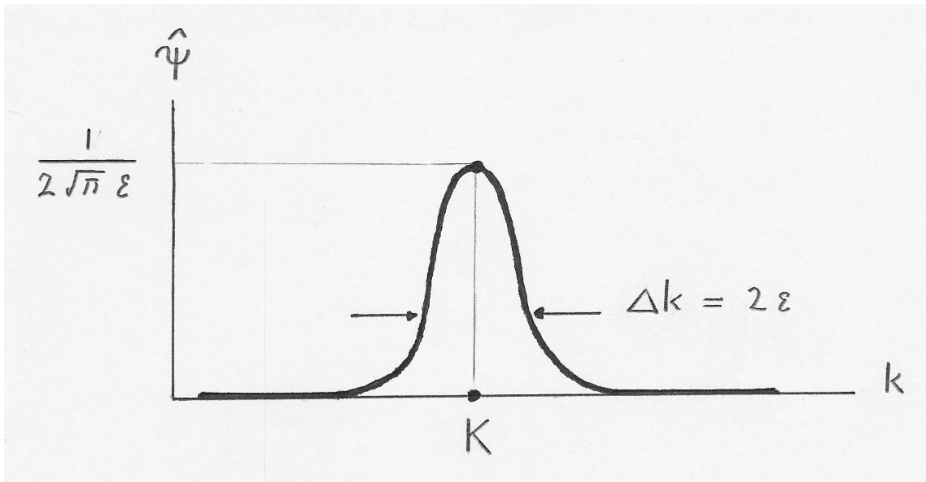

Figure 4.7

Notice how the change of variable nicely separates out the carrier wave e^{iKx}. The remaining integral succumbs to a famous trick, called "completing the square": Set $a := i\varepsilon x$ and write the exponent in (4.22) as

$$-u^2 + 2a\kappa - a^2 + a^2 = -(u-a)^2 + a^2,$$

so

$$\int_{-\infty}^{\infty} e^{-u^2+2au} du = e^{a^2} \int_{-\infty}^{\infty} e^{-(u+a)^2} du. \tag{4.23}$$

If a were real, we'd say that "the origin of u does not matter, and the area under the Gaussian $e^{-(u-a)^2}$ is $\sqrt{\pi}$, independent of a". We'd conclude that the right-hand side of (4.23) is $\sqrt{\pi}e^{a^2}$. But $a := i\varepsilon x$ is *pure imaginary*. Here is another instance in which a result from real variable calculus extends to the complex plane: We have

$$\int_{-\infty}^{\infty} e^{-u^2+2au} du = \sqrt{\pi}e^{a^2} \tag{4.24}$$

for *all* complex a. An exercise outlines the plausibility based on contour integration. From (4.24) with $a = i\varepsilon x$ and (4.22) it follows that

$$\psi(x,0) = \mathrm{Re}\, e^{iKx} e^{-(\varepsilon x)^2} = e^{-(\varepsilon x)^2} \cos Kx. \tag{4.25}$$

This wave packet has the same carrier wave $\cos Kx$ as (4.18), but a Gaussian envelope $c^{-(\varepsilon x)^2}$. We take the width of this envelope to be $\Delta x = \frac{2}{\varepsilon}$, corresponding to the interval of x where $e^{-(\varepsilon x)^2} > \frac{1}{c}$. For the wave packet based on Gaussian spectrum, we have $\Delta x \Delta k = 4$ independent of ε. What is important here is the independence from ε. The numerical prefactor (4π for the "skyscraper" spectrum, or 4 for the Gaussian) depends on (unimportant) details of how Δx and Δk are defined.

We now "turn on time" and examine how the wavefield (4.14) moves. We'll work out the case of the "skyscraper" spectrum with $\hat{\psi}(k)$ given by (4.15). Then (4.14) reads

$$\psi(x,t) = \mathrm{Re}\,\frac{1}{2\varepsilon}\int_{K-\varepsilon}^{K+\varepsilon} e^{i(kx-\omega(k)t)}dk.$$

Changing the integration variable to u in (4.21) helps. We find

$$\psi(x,t) = \mathrm{Re}\,\frac{e^{iKx}}{2}\int_{-1}^{1} e^{i\varepsilon xu-\omega(K+\varepsilon u)t}du. \tag{4.26}$$

In the limit $\varepsilon \to 0$, we employ the first order Taylor polynomial of $\omega(K+\varepsilon u)$ in ε,

$$\omega(K-\varepsilon u) = \omega(K) - \varepsilon u\omega'(K) + O(\varepsilon^2)$$

and (4.26) becomes

$$\psi(x,t) = \mathrm{Re}\,e^{i(Kx-\omega(K)t)}\frac{1}{2}\int_{-1}^{1} e^{i\varepsilon(x-\omega'(K)t)+O(\varepsilon^2)}du.$$

We see the carrier wave nicely separated out. We approximate the remaining integral by ignoring the $O(\varepsilon^2)$ truncation error, and we get

$$\psi(x,t) = \cos(Kx-\omega(K)t)\frac{\sin\varepsilon(x-\omega'(K)t)}{\varepsilon(x-\omega'(K)t)}\,. \tag{4.27}$$

We see that the envelope retains the same shape as it has at time zero, and *translates* at the group velocity $\omega'(K)$.

Complex exponential solutions of ODE

Figure 4.8 shows an elastic rod suspended from the ceiling by an array of closely spaced spring-like threads. The elevation of the rod relative to its

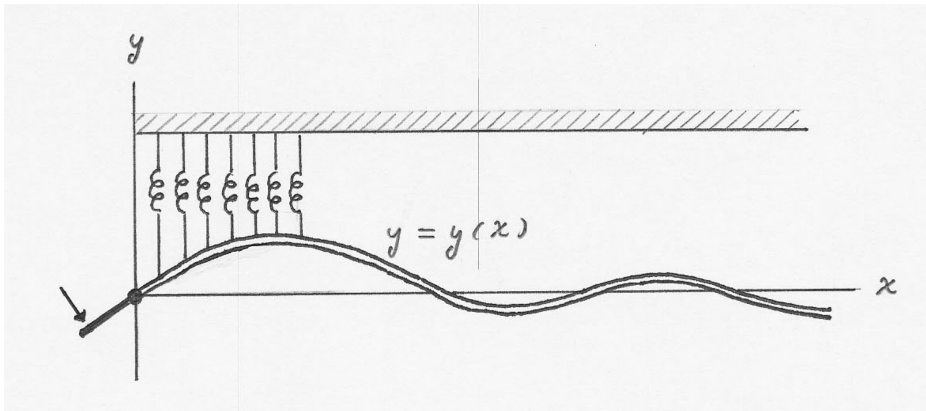

Figure 4.8

rest configuration is denoted by $y = y(x)$ in $x \geq 0$. The local mechanical equilibrium of the rod is expressed by the (dimensionless) ODE

$$y^{(4)}(x) = -y(x). \tag{4.28}$$

Here $y^{(4)}(x)$ is the vertical force per unit length that must be applied to the rod to maintain a given shape $y = y(x)$.[1] For the hanging rod in Figure 4.8, this force is provided by the spring-like threads and gravity. Acting together they provide a net restoring force $-y$ per unit length toward the x-axis.

The ODE (4.28) is linear and *translation invariant* in x. Translation invariance means that if $y(x)$ is a solution, so is $y(x + \varepsilon)$ for any constant ε.

[1]This fourth derivative "bending force" derives from the *principal of virtual work*: The elastic potential energy of the rod due to bending is

$$e = \frac{1}{2}\int_0^\infty (y''(x))^2 dx.$$

Here, the second derivative y'' represents local bending, and $\frac{1}{2}(y'')^2$ represents the local bending energy per unit length. Suppose we change the rod configuration from $y(x)$ to $y(x) + (dy)(x)$. The resulting differential of e is

$$de = \int_0^\infty y''(x)(dy)''dx = \int_0^\infty y^{(4)}(x)(dy)(x)dx.$$

The last equality is two integrations by parts, assuming dy and $(dy)'$ vanish at $x = 0$ and $x = \infty$. de is the work done in the deflection from $y(x)$ to $y(x) + (dy)(x)$, so $y^{(4)}(x)$ is the force per unit length that is applied.

Such ODE have *elementary solutions* in the form of exponentials

$$y = e^{zx} \tag{4.29}$$

where z is a constant, to be determined. Substituting (4.29) into ODE (4.28), we find that z satisfies the *characteristic equation*

$$z^4 = -1.$$

Hence, z is one of the fourth roots of -1, depicted in Figure 4.9. The

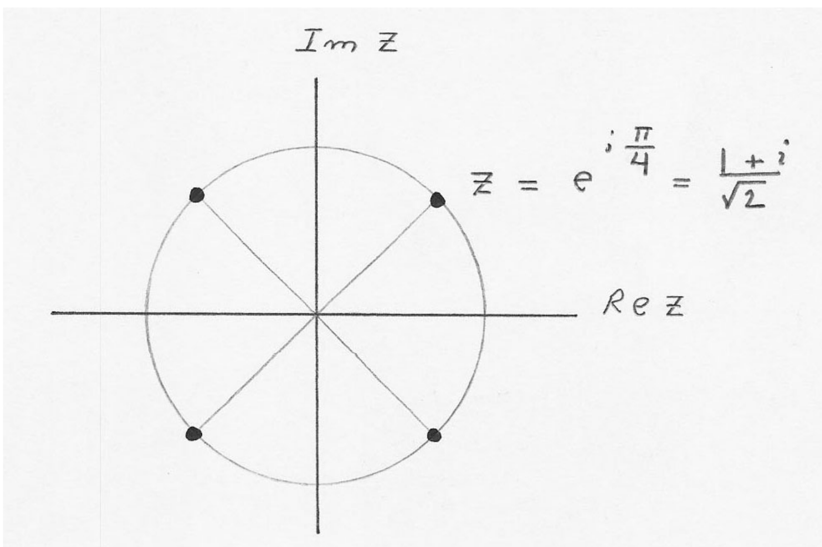

Figure 4.9

exponential solution corresponding to $z = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$ is

$$e^{\frac{x}{\sqrt{2}}} e^{i\frac{x}{\sqrt{2}}}. \tag{4.30}$$

The real and imaginary parts of a complex solution are each real solutions in their own right, hence we obtain from (4.30) real solutions

$$e^{\frac{x}{\sqrt{2}}} \cos \frac{x}{\sqrt{2}}, \; e^{\frac{x}{\sqrt{2}}} \sin \frac{x}{\sqrt{2}}\,. \tag{4.31}$$

The ODE (4.28) is invariant under "x-reversal", meaning that $y(-x)$ is a solution if $y(x)$ is. Hence, we obtain two more solutions from (4.31),

$$e^{-\frac{x}{\sqrt{2}}}\cos\frac{x}{\sqrt{2}},\ e^{-\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}. \tag{4.32}$$

The *general solution* of the fourth order ODE (4.28) consists of all linear combinations of the four elementary solutions in (4.31), (4.32).

We examine the rod configuration that results from "pivoting the rod" at $x = 0$. The rod is "pinned" to $(0, 0)$, but we can tilt it from the horizontal by an attached lever as depicted in Figure 4.8. Hence, the boundary conditions at $x = 0$ are $y(0) = 0$, $y'(0) = m > 0$. As $x \to +\infty$, $y(x)$ presumably asymptotes to the rest configuration $y = 0$. The zero boundary conditions at $x = 0, \infty$ single out solutions proportional to

$$e^{-\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}$$

and the solution with $y'(0) = m$ is

$$y(x) = \sqrt{2}me^{-\frac{x}{\sqrt{2}}}\sin\frac{x}{\sqrt{2}}. \tag{4.33}$$

Figure 4.8 depicts this configuration, with its *oscillatory* decay to zero as $x \to +\infty$.

What is the torque required to maintain the tilt $y'(0) = m > 0$? The total work to tilt the rod from $y'(0) = 0$ to $y(0) = m > 0$ is equal to the potential energy

$$u = \frac{1}{2}\int_0^\infty \{(y'')^2 + y^2\}dx. \tag{4.34}$$

Here $\frac{1}{2}(y'')^2$ is the bending energy per unit length and $\frac{1}{2}y^2$ is associated with the restoring force $-y$ due to joint action of the spring-like threads and gravity. Substituting the solution (4.33) for $y(x)$ into (4.34), we obtain u as a function of m, and the torque (for small m) is $\frac{du}{dm}$. This calculation, and a deeper investigation of a "torque balance" boundary condition at $x = 0$ are presented in an exercise.

Resonance

The analysis of mechanical and electrical networks often leads to linear differential equations with constant coefficients subject to sinusoidal forcing. A most fundamental example is the damped, forced harmonic oscillator. The (dimensionless) ODE is

$$\ddot{x} + \varepsilon\dot{x} + x = \cos\omega t, \tag{4.35}$$

where ε is a dimensionless damping coefficient and ω is the angular frequency of the forcing, relative to the natural frequency. Any solution $x(t)$ of (4.35) asymptotes to a unique periodic solution as $t \to \infty$. The periodic solution takes the form

$$x(t) = R(\omega)\cos(\omega t + \theta(\omega)) \tag{4.36}$$

where the *amplitude* $R(\omega)$ and *phase shift* $\theta(\omega)$ are functions of ω to be determined from the ODE (4.35).

While it is possible to substitute (4.36) into (4.35) and derive equations for $R(\omega)$ and $\theta(\omega)$ by massive use of trig identities, the streamlined analysis based on the complex exponential is vastly better. First, we replace (4.35) by the ODE

$$\ddot{z} + \varepsilon\dot{z} + z = e^{i\omega t} \tag{4.37}$$

for the *complex valued* function of time, $z(t)$. Given $z(t)$, $x(t) := \operatorname{Re} z(t)$ satisfies (4.35). Next, we compute the solution of (4.37) proportional to $e^{i\omega t}$,

$$z(t) = a(\omega)e^{i\omega t} \tag{4.38}$$

where $a = a(\omega)$ is a *complex* amplitude. Substituting (4.38) into (4.37) we obtain a formula $a(\omega)$,

$$(-\omega^2 + i\varepsilon\omega + 1)a = 1,$$

or

$$a(\omega) = \frac{1}{1 - \omega^2 + i\varepsilon\omega}\,. \tag{4.39}$$

If we express a in polar form,

$$a = \mathrm{Re}^{i\theta}$$

then $z = \mathrm{R}e^{i(\omega t+\theta)}$ and $x = R\cos(\omega t+\theta)$, so presto bingo we see that R and θ in (4.36) are precisely the modulus and argument of the complex amplitude a. The amplitude R is

$$R(\omega) = |a| = \frac{1}{\sqrt{(1-\omega^2)^2 + \varepsilon^2\omega^2}}\,. \tag{4.40}$$

The graph of $R(\omega)$ for $\omega > 0$ and $0 < \varepsilon \ll 1$ is depicted in Figure 4.10: The

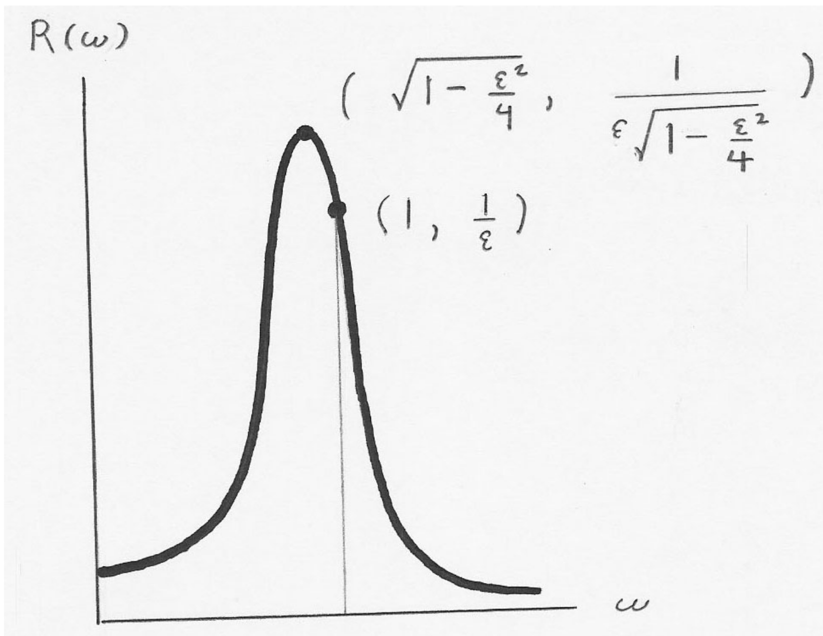

Figure 4.10

sharp peak of $R(\omega)$ near $\omega = 1$ is called *resonance*. The examination of the phase shift $\theta(\omega)$ gives some mechanical intuition of resonance. There is a nice geometric description of $\theta(\omega)$: First, write (4.39) as

$$a = \frac{1-\omega^2 - i\varepsilon\omega}{(1-\omega^2)^2 + \varepsilon^2\omega^2}\,,$$

so

$$\theta = \arg(\zeta(\omega)) \tag{4.41}$$

where

$$\zeta(\omega) := 1 - \omega^2 - 2i\varepsilon\omega. \tag{4.42}$$

(4.42) is the parametric representation of a parabola in the complex plane, depicted in Figure 4.11. This parabola is oriented in the direction of increas-

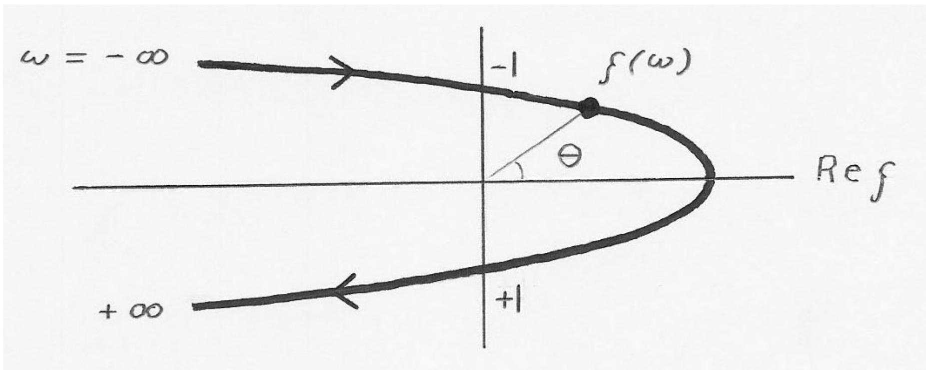

Figure 4.11

ing ω. As ω increases from zero to one, we see that θ decreases from 0 at $\omega = 0$ to $-\pi$ as $\omega \to +\infty$, passing through $\theta = -\frac{\pi}{2}$ at $\omega = 1$. For $0 < \varepsilon \ll 1$, $\omega = 1$ is very close to the resonance peak at $\omega = \sqrt{1 - \frac{\varepsilon^2}{2}}$ (see Figure 4.10). The phase shift $\theta(1) = -\frac{\pi}{2}$ indicates that the *x-oscillation lags behind the forcing by one quarter of a period.* This is depicted in Figure 4.12a. Here

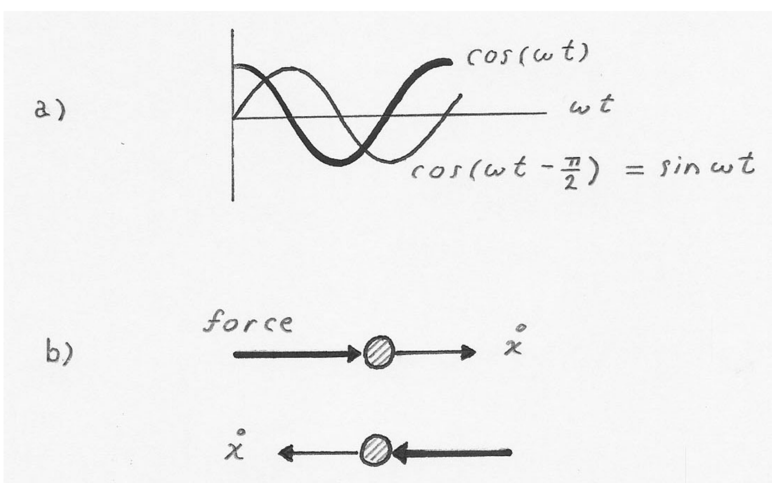

Figure 4.12

is what this phase lag means physically: When the oscillator (that is $x(t)$)

is passing through $x = 0$ with the maximum speed, the force is also at its maximum, and pushes in the same direction as the velocity. This is depicted in Figure 4.12b. You may have experienced the pleasure of shaking a stop sign pole back and forth. When you create a juicy resonance, you can actually feel how your maximum push or pull happens when the pole is in the middle of a swing, in the upright configuration, and moving the fastest. Your applied force at these times is in the same direction as the motion.

AC electrical networks

Figure 4.13a depicts a resistor, connected at one end to a voltage supply,

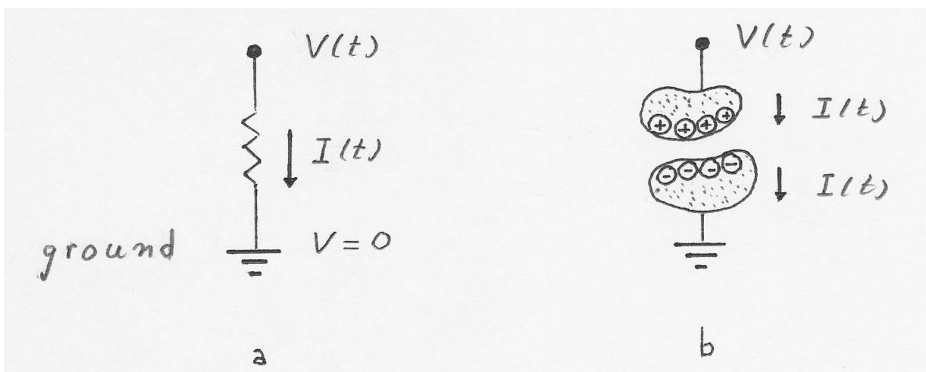

Figure 4.13

and the other, to *ground* ("ground" means voltage zero). The electric current flowing through the resistor is proportional to the voltage drop across it, according to Ohm's law

$$V(t) = RI(t). \tag{4.43}$$

The positive proportionality constant R is called the *resistance*. Figure 4.13b depicts the simplest *capacitor*, consisting of two conducting electrodes one connected to the voltage supply, the other to ground, and insulated from each other. When the voltage supply induces charge $Q(t)$ on the electrode connected to it, an opposite and equal charge $-Q(t)$ is induced on the electrode connected to ground. The voltage drop from the $+Q$ to $-Q$ electrodes is proportional to V according to

$$Q(t) = CV(t). \tag{4.44}$$

The positive constant C is called the *capacitance*. Taking the time derivative of ( ) gives

$$\dot{V}(t) = CI(t). \tag{4.45}$$

Here $I(t) := \dot{Q}(t)$ is the electric current into the $+Q$ electrode. Since charges of opposite sign are entering the $-Q$ electrode at the same rate, the electric current from the $-Q$ electrode to ground is also $I(t)$.

We examine the situation of "alternating current" (AC) in which $V(t)$ and $I(t)$ are sinusoidal in time. As in the analysis of the harmonic oscillator, we introduce formal complex-valued voltages and currents expressed as complex exponentials. Physical voltages and currents are extracted from real parts. That is the voltage $V(t)$ and current $I(t)$ are taken to be

$$V(t) = Ve^{i\omega t}, \; I(t) = Ie^{i\omega t}, \tag{4.46}$$

where V and I are *complex voltage and current amplitudes*. From the current-voltage relations (4.43), (4.45) for resistor and capacitor, we deduce linear relations between voltage and current amplitudes,

$$\begin{aligned} V &= RI \text{ (resistor)}, \\ V &= \frac{I}{i\omega C} \text{ (capacitor)}. \end{aligned} \tag{4.47}$$

Resistors and capacitors are examples of *linear* devices. A linear device is characterized by a complex *impedance* $Z = Z(\omega)$, so that voltage and current amplitudes satisfy

$$V = Z(\omega)I. \tag{4.48}$$

For instance, look at a resistor and capacitor in parallel, as in Figure 4.14. Given V, the complex current amplitudes are $\frac{V}{R}$ and $i\omega CV$, and the ampli-

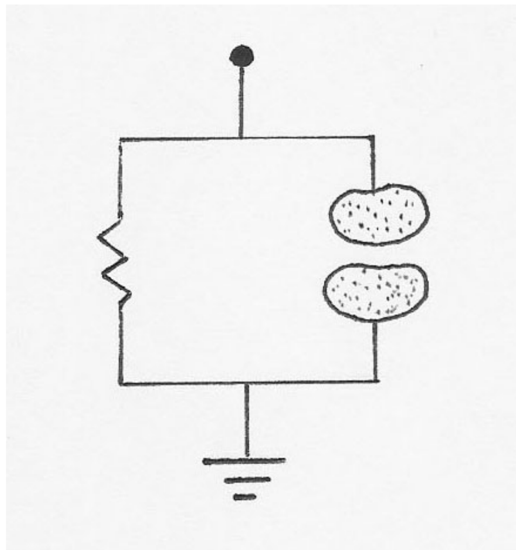

Figure 4.14

tude I of the total current from the voltage supply to ground is the sum

$$I = \left(\frac{1}{R} + i\omega C\right) V. \tag{4.49}$$

Hence the impedance of the resistor and capacitor in parallel is $Z(\omega)$ which satisfies

$$\frac{1}{Z(\omega)} = \frac{1}{R} + i\omega C. \tag{4.50}$$

In general, two devices in parallel as in Figure 1.15a, have "composite"

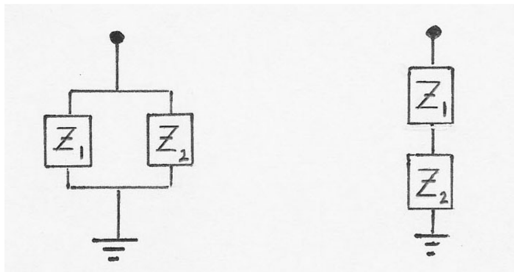

Figure 4.15

impedance Z which satisfies

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}. \tag{4.51}$$

For the two devices in series as in Figure 1.15b, the current through each device is the same, and the voltage drops add, and this leads to the composite impedance which is the sum

$$Z = Z_1 + Z_2. \tag{4.52}$$

Frequency dependent response

The impedance as a function of frequency quantifies the amplitude and phase of the current relative to voltage. First, write (4.48) as

$$I = \frac{V}{Z(\omega)} = A(\omega)e^{i\theta(\omega)}V, \tag{4.53}$$

where $A(\omega)$ and $\theta(\omega)$ are the modulus and argument of $\frac{1}{Z(\omega)}$. Take V to be real. Then the physical voltage and current as functions of time are

$$\begin{aligned} \mathrm{Re}(Ve^{i\omega t}) &= V\cos(\omega t), \\ \mathrm{Re}(Ie^{i\omega t}) &- A(\omega)V\cos(\omega t + \theta(\omega)). \end{aligned} \tag{4.54}$$

We se that $D(\omega)$ gives the magnitude of the current oscillation, relative to the voltage oscillation, and $\theta(\omega)$ represents a phase shift of current relative

Figure 4.16

to voltage (phase *lag* if $\theta < 0$, phase *advance* if $\theta > 0$). For instance, consider the resistor and capacitor in parallel, as in Figure 4.14, which has $\frac{1}{Z(\omega)}$ as in (4.49). Figure 4.16 depicts $\frac{1}{Z(\omega)}$ as a vertical line ray in the complex plane parametrized by $\omega > 0$. We immediately see that

$$\begin{aligned} A(\omega) &= \sqrt{\frac{1}{R^2} + \omega^2 C^2} = \frac{1}{R}\sqrt{1 + (\omega RC)^2}, \\ \theta(\omega) &= \arctan(\omega RC). \end{aligned} \tag{4.55}$$

Figure 4.17 shows the graphs of $A(\omega)$ and $\theta(\omega)$. As $\omega RC \to 0$, we have $A \to \frac{1}{R}$, $\theta \to 0$, so the circuit becomes a pure resistor in the low frequency limit. This is because most of the current goes through the resistor in the low frequency limit. In the high frequency limit $\omega RC \to \infty$, most of the current goes through the capacitor, and we have $\frac{A}{\omega C} \to 1$, $\theta \to \frac{\pi}{2}$ which corresponds to pure capacitance.

Figure 4.17