Chapter 3

Complex variables

Complex numbers begin with the notion, that all quadratic equations with real coefficients "ought" to have solutions. The quadratic equation $x^2+1=0$ has no real solution, but since it "ought" to have a solution we'll just say that there is "some sort of number", denoted i, whose square is -1:

$$i^2 = -1. \tag{3.1}$$

Formal linear combinations $a + bi$ with a, b real arise naturally as formal solutions of more general quadratic equations. For instance, the quadratic equation

$$x^2 - 4x + 13 = 0$$

is equivalent to

$$\left(\frac{x-2}{3}\right)^2 + 1 = 0,$$

and formally, we have $\frac{x-2}{3} = \pm i$, so $x = 2 \pm 3i$.

The usual heuristic introduction to complex numbers begins like this: "Repesent a complex number as $z = a + bi$, with a, b real. You add and multipy complex numbers, as if the usual laws of arithmetic hold, with the additional feature, $i^2 = -1$." The formal sum of two complex numbers can be arranged into another complex number:

$$(a + bi) + (c + di) = (a + c) + (bi + di) = a + c + (b + d)i.$$

The first equality uses commutative and associate laws of addition, and the second, the distributive law. For complex multiplication, we have

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Since the sum and product of complex numbers are complex numbers, we say that the complex numbers are *closed under addition and multiplication*. Apparently we don't need to enlarge the complex numbers beyond the set of $a + bi$ with a, b real.

The initial heuristics informs the official definition: The set $\mathbb{C}$ of *complex numbers* consists of ordered pairs of real numbers

$$z = (a, b) \tag{3.2}$$

subject to binary operations of *addition* and *multiplication*, defined by

$$(a, b) + (c, d) = (a + c, b + d) \tag{3.3}$$

$$(a, b)(c, d) = (ac - bd, ad + bc). \tag{3.4}$$

In (3.2), $\operatorname{Re} z := a$ and $\operatorname{Im} z := b$ are called *real* and *imaginary*[1] parts of the complex number z. We review the basic arithmetic and geometry of complex numbers within the framework of the official definition.

- The definitions (3.3), (3.4) of complex addition and multiplication, and the 'laws' of real arithmetic (commutative and associative laws of addition and multiplication, distributive laws) imply that the same arithmetic laws extend to the complex numbers.

- $z = (a, 0)$ corresponds to the real number a. By the multiplication rule (3.4), we have
  $$(0, 1)^2 = (0, 1)(0, 1) = (-1, 0).$$
  Since $(-1, 0)$ corresponds to -1, we identify $(0, 1)$ with i,
  $$i = (0, 1).$$
  The general complex number in (3.2) can be represented as
  $$z = (a, b) = (a, 0) + (b, 0)(0, 1),$$
  which corresponds to the traditional notation $z = a + bi$. Henceforth we revert to the traditional notation.

- The complex number $z = x + yi$ is represented geometrically as a point in the plane with cartesian coordinates x and y, as in Figure 3.1. We call this one-to-one correspondence between complex numbers and points in the plane the *complex plane*.

[1] imaginary, as in "we just imagined them".

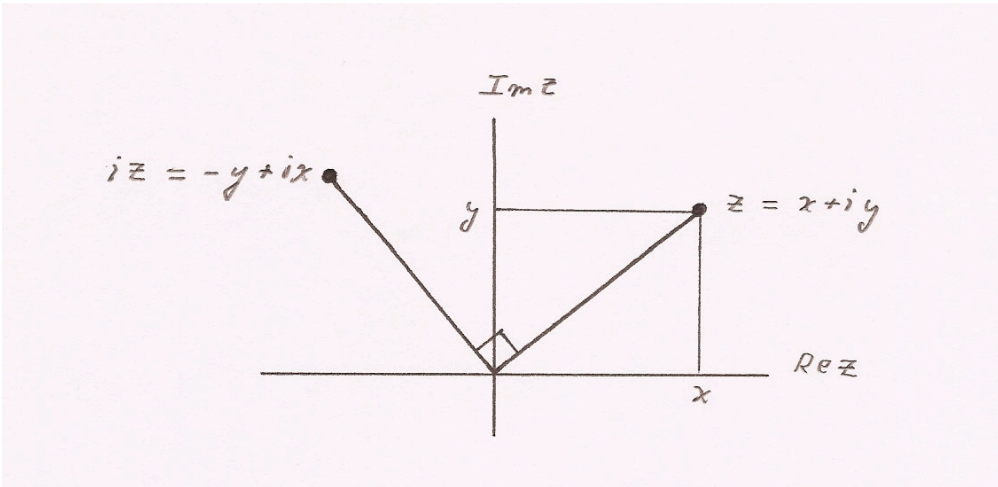

Figure 3.1

Given $z = x + yi$ we have $iz = -y + xi$. Geometrically, multiplication by i means rotation by $\frac{\pi}{2}$ counterclockwise radians. This is visualized in Figure 3.1. Multiplication by i^2 represents rotation by π radians, and rotation of z by π radians produces $-z$. We calculate $i^2 z = i(iz) = i(-y + ix) = -i - iy = -z$. The mysterious identity $i^2 = -1$ has been deconstructed into: "Two successive 'left faces' equals one 'about face'."

In the real number system, zero and one are distinguished as *additive* and *multiplicative identities*, and they retain these roles in the complex number system. For any complex number z, $z + 0 = z$, and $z \cdot 1 = z$. $z = a + ib$ has the unique *additive inverse* $-z := -a + (-b)i$, so $z - z := z + (-z) = 0$. For $z \neq 0$, there is a unique *multiplicative inverse* z^{-1} which satisfies $zz^{-1} = 1$. Setting $z^{-1} = \alpha + \beta i$, we have $zz^{-1} = a\alpha - b\beta + (b\alpha + a\beta)i = 1 = 1 + 0i$, and equating real and imaginary parts,

$$\begin{aligned} a\alpha - b\beta &= 1, \\ b\alpha + a\beta &= 0. \end{aligned} \tag{3.5}$$

The determinant of this linear system for α, β is $a^2 + b^2 \neq 0$, since $z = a + ib \neq 0$ means "at least one of a or b non-zero". Geometrically, $|z| := \sqrt{a^2 + b^2}$, called the *modulus* of z, is the length of displacement from $(0, 0)$ to (a, b) in the complex plane. The solution of (3.5) for α, β is $\alpha = \frac{a}{a^2+b^2}$, $\beta = \frac{-b}{a^2+b^2}$, so

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\overline{z}}{|z|^2} \,. \tag{3.6}$$

Here,

$$\overline{z} := a - bi \tag{3.7}$$

is called the *complex conjugate* or conjugate of z. Geometrically, conjugation of z is reflection about the real axis. Figure 3.2 depicts the geometric mean-

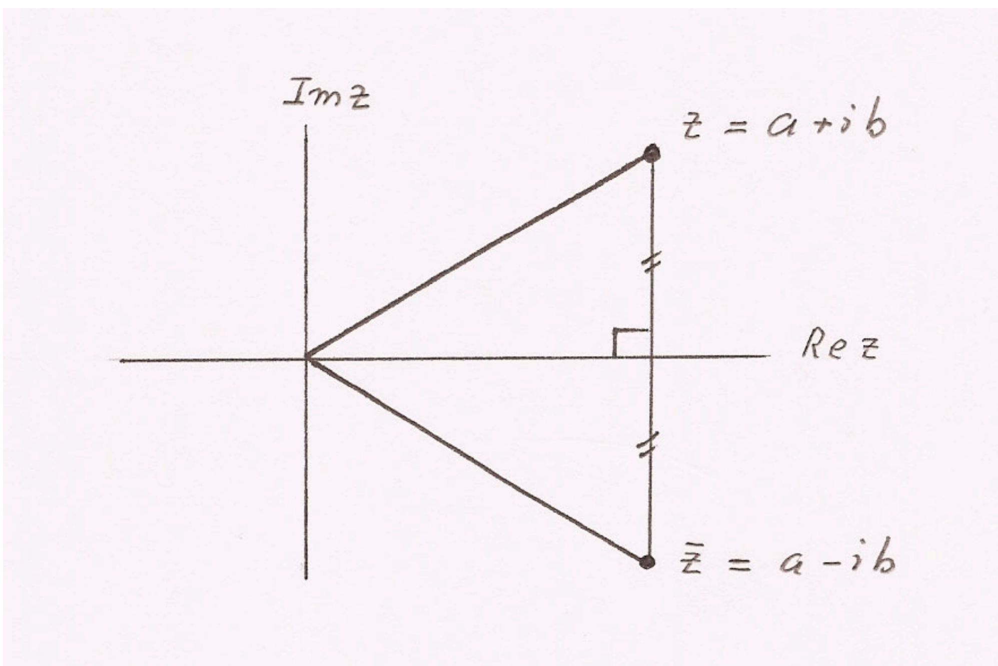

Figure 3.2

ings of modulus and conjugate. As in real arithmetic, “z_1 divided by $z_2 \neq 0$ means $(z_1)(z_2^{-1})$ and we'll denote it $\frac{z_1}{z_2}$ just like in real arithmetic.

There are simple properties of conjugation that you need to relate to, like a fish relates to water. They are

$$
\begin{aligned}
z\overline{z} &= |z|^2, \\
\overline{\overline{z}} &= z, \\
\overline{z_1 + z_2} &= \overline{z}_1 + \overline{z}_2, \\
\overline{z_1 z_2} &= \overline{z}_1 \overline{z}_2, \\
\overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\overline{z}_1}{\overline{z}_2},\ z_2 \neq 0.
\end{aligned}
\tag{3.7}
$$

The last three identities are a rare instance for which "mindless manipulation of symbols" actually works.

Here is a typical example of the roles of conjugate and modulus in routine algebraic calculations: We want to find the real and imaginary parts of $\left(\frac{1+i}{1-i}\right)^3$. We calculate

$$
\left(\frac{1+i}{1-i}\right)^3 = (1+i)^3\left(\frac{1}{1-i}\right)^3 = (1+i)^3\left(\frac{1+i}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}}(1+i)^6. \tag{3.8}
$$

Next, binomial expansion:

$$
\begin{aligned}
(1+i)^6 &= 1 + 6i + 15i^2 + 20i^3 + 15i^4 + 6i^5 + i^6 \\
&= (1 - 15 + 15 - 1) + (6 - 20 + 6)i = -8i.
\end{aligned}
\tag{3.9}
$$

We used the sixth row of Pascal triangle, and $i^2 = -1$, $i^3 = -i$, etc. Combining (3.8), (3.9) we have

$$
\left(\frac{1+i}{1-i}\right)^3 = -\frac{4}{\sqrt{2}}i.
$$

Properties of the modulus which are corollaries of conjugation identities (3.7) are

$$
|z_1 z_2| = |z_1|\,|z_2|
$$

and

$$
\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0.
$$

Notice that $|z_1 + z_2| \neq |z_1| + |z_2|$. By visualizing $z_1 + z_2$ as vector addition of z_1 and z_2 in the complex plane

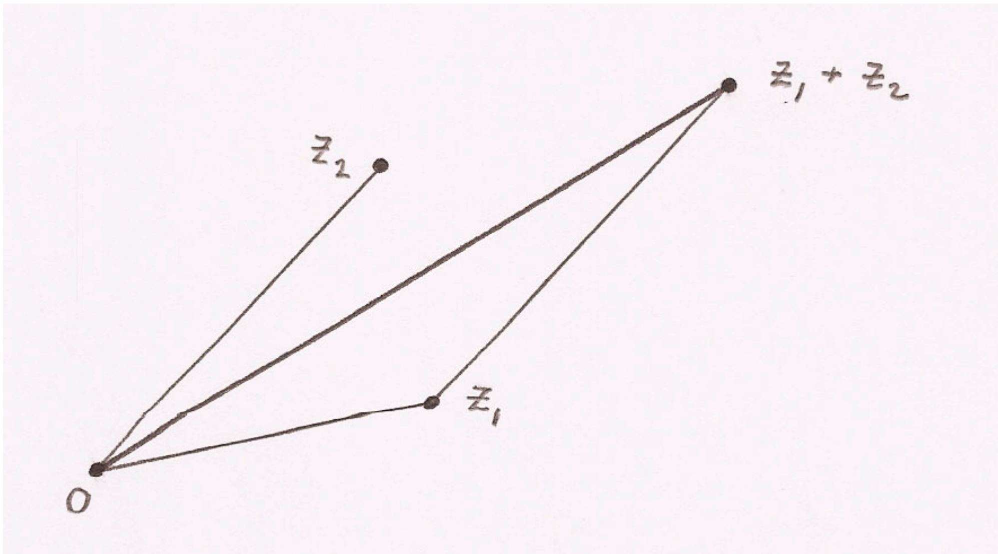

Figure 3.3

we discern the triangle inequality,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

As a challenge for you: What geometric picture informs the inequality

$$|z_1 - z_2| > ||z_1| - |z_2||?$$

An essential connection between the algebra and geometry of complex numbers is revealed by polar forms of complex multiplication and division. The *polar form* of the complex number $z = x + yi$ results by introducing polar coordinates r, θ of the point (x, y) as depicted in Figure 3.4. The polar form of z is

$$z = r(\cos\theta + i\sin\theta). \tag{3.10}$$

Given z, $r = |z|$ is uniquely determined. Not so the angle θ: For any integer k we can replace θ by $\theta + 2\pi k$ in (3.10) and obtain the same z. For instance, (3.10) with $r = 1$, and $\theta = \frac{\pi}{4}$ or $-\frac{7\pi}{4}$ are both polar representations of $1 + i$. This is depicted in Figure 3.5. Sometimes we like to think of the angle θ as a function of $z \neq 0$. Any one of the possible angles θ associated with a given $z \neq 0$ is called an *argument of* z, denoted $\theta = \arg z$. The multivalued character of $\arg z$ is displayed in the "spiral ramp" surface in Figure 3.6. The

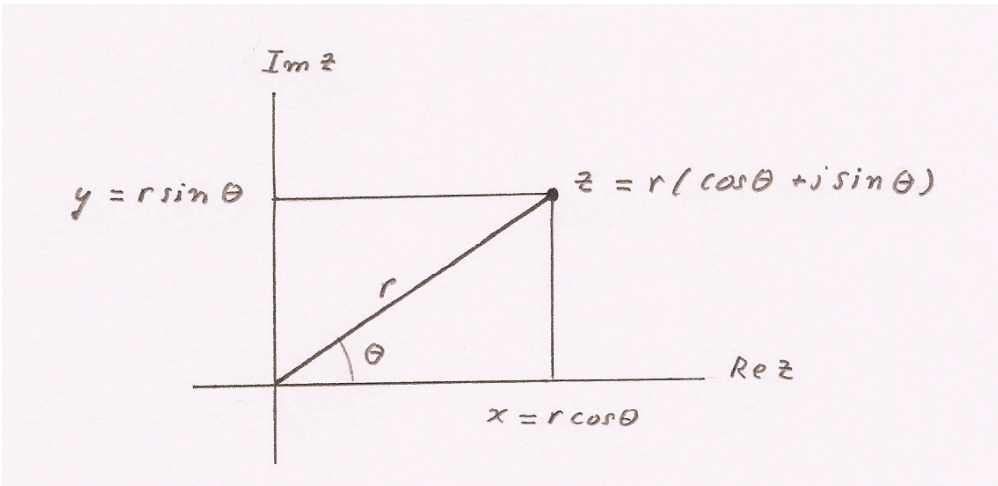

Figure 3.4

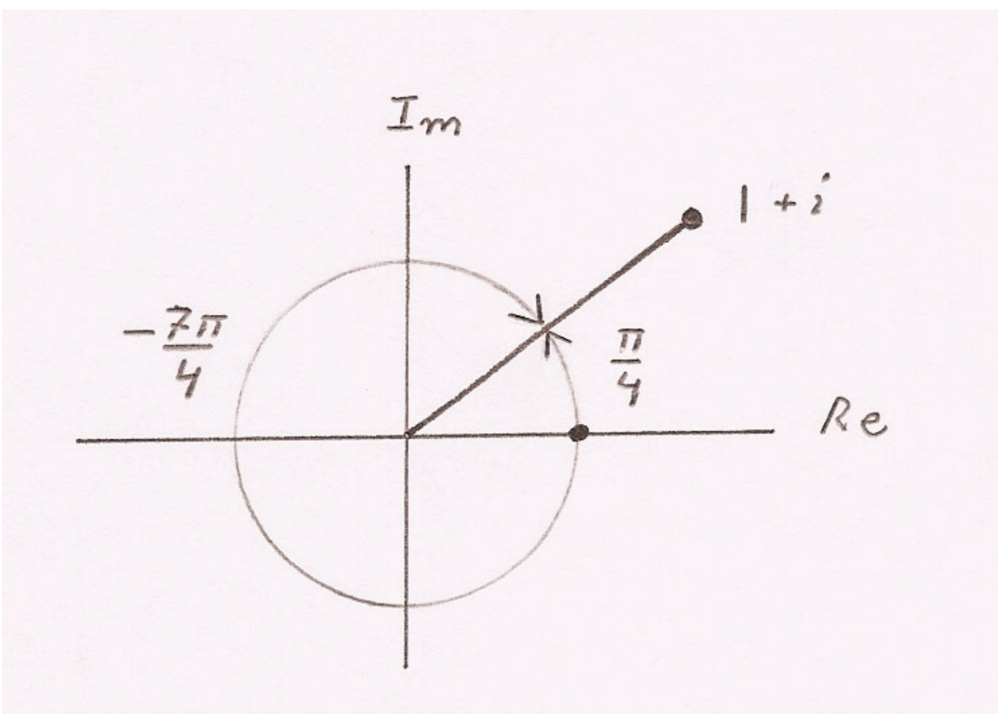

Figure 3.5

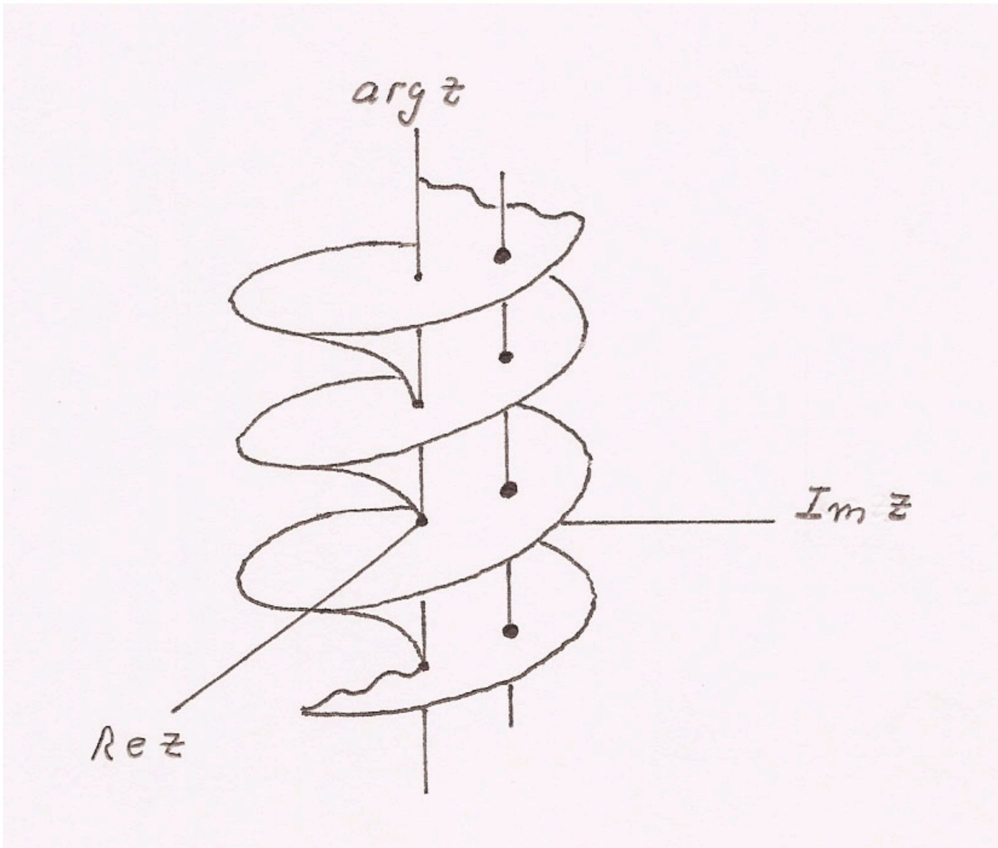

Figure 3.6

metal shavings produced by drilling a large hole with a well-honed bit can often look like this. A given value of $z \neq 0$ is represented by a vertical line. Its intersections with the spiral ramp correspond to the values of $\arg z$.

Given polar forms of complex numbers z_1 and z_2, the polar form of the product is computed from

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}. \end{aligned} \tag{3.11}$$

Geometrically, complex multiplication amounts to multiplying moduli, and adding arguments (that is, angles). If $r_2 \neq 0$, we find that the polar form of quotient $\frac{z_1}{z_2}$ has modulus $\frac{r_1}{r_2}$ and its argument is the difference of arguments $\theta_1 - \theta_2$.

As a first simple exercise, let's reconsider the calculation (3.9). The polar version: We have $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so $(1+i)^6 = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -8i$, and the brutal binomial expansion is nicely side-stepped.

The next example leads into a most important application of the polar form: Let $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then $z^3 = \cos 2\pi + i\sin 2\pi = 1 + 0i = 1$, so z is evidently a complex cube root of one, in addition to the usual real cube root 1. By conjugation properties $z^3 = 1$ implies $\overline{z}^3 = 1$ as well, so we have *three* cube roots of one, equally spaced around the unit circle. The general construction of complex n-th roots z of any complex number w

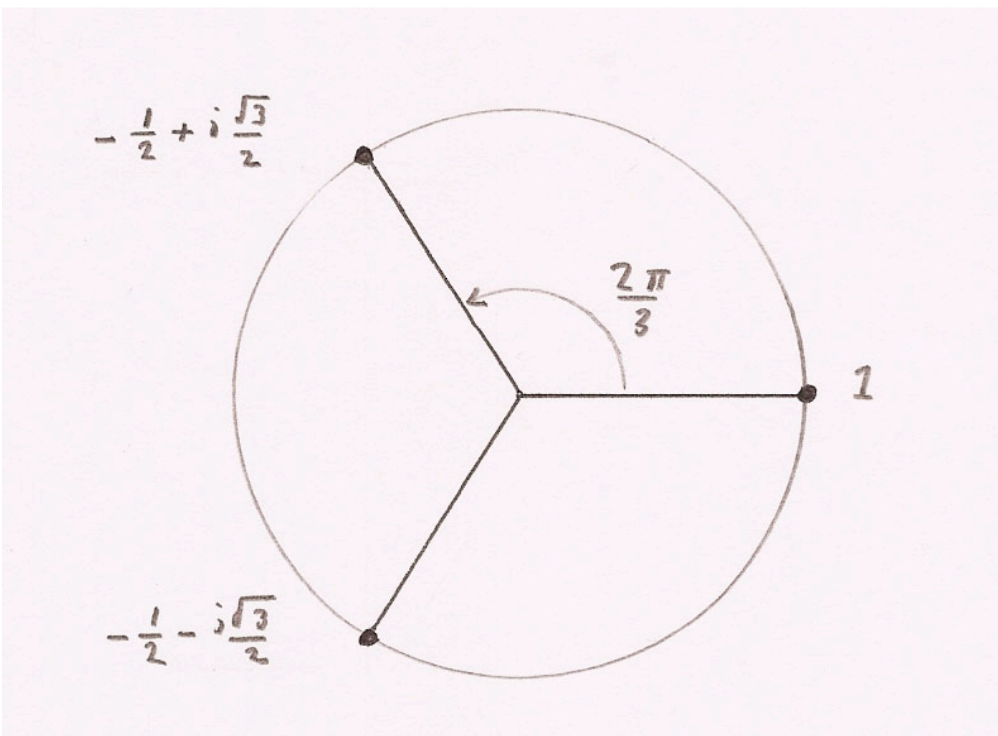

Figure 3.7

goes like this: Put w in polar form $w = \rho(\cos\varphi + i\sin\varphi)$, and seek n-th roots in polar form, $z = r(\cos\theta + i\sin\theta)$. We have

$$\begin{aligned} z^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= \rho(\cos\varphi + i\sin\varphi) = w. \end{aligned} \tag{3.12}$$

The moduli on both sides are equal, so

$$r = \rho^{\frac{1}{n}},$$

where the right-hand side is the usual positive n-th root of ρ. Next, observe that (3.12) with $r^n = \rho$ holds if the angles $n\theta$ and φ are equal, and *also if* φ *and* $n\theta$ *differ by an integer multiple of* 2π, so

$$\begin{aligned} n\theta &= \varphi + 2\pi k, \text{ or} \\ \theta &= \frac{\varphi}{n} + \frac{2\pi}{n}k, \end{aligned}$$

where k is an integer. In summary, we have n-th roots of $w = \rho(\cos\varphi + i\sin\varphi)$ given by

$$z = \rho^{\frac{1}{n}}\left\{\cos\left(\frac{\varphi}{n} + \frac{2\pi}{n}k\right) + i\sin\left(\frac{\varphi}{n} + \frac{2\pi}{n}k\right)\right\}, \tag{3.13}$$

where k is any integer. In (3.13), only $k = 0, 1, \ldots n-1$ yield distinct n-th roots z. All other integers k just repeat one of these n values. Figure 3.8 visualizes the construction of n-th roots in (3.13), for $n = 6$. The construction of complex n-th roots, and more generally, the complex solutions of n-th degree polynomial equations, play essential roles in constructing solutions to linear ODE and PDE that routinely arise in physical applications.

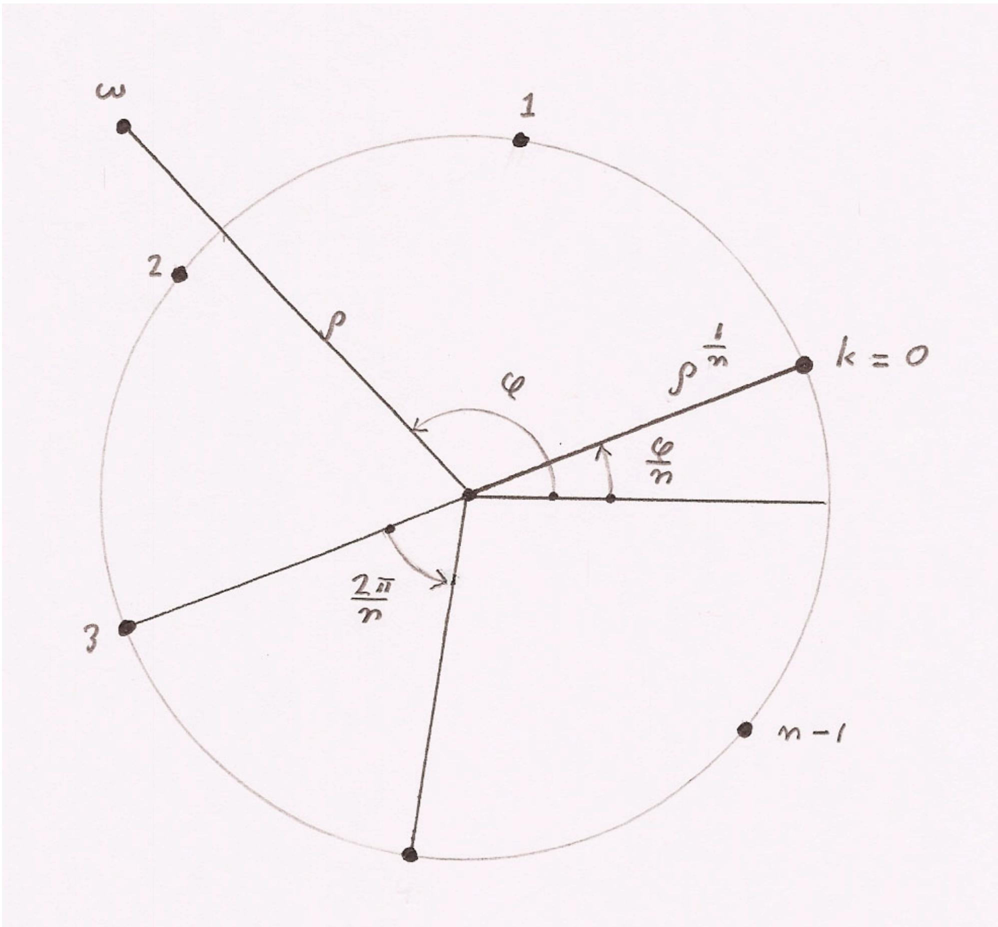

Figure 3.8

Complex functions and their power series

There is a whole calculus of complex functions of a complex variable which generalizes the usual calculus of functions of a real variable. This chapter sets forth some essentials of this calculus which routinely arise in solutions of ODE and PDE.

First, we recognize that complex functions of a complex variable are much richer objects than real functions of a real variable. For each z in a region D of the complex plane, the function f assigns a complex number $w = f(z)$.

As z "paints" the region D, the corresponding w's typically "paint" a region $f(D)$ in the complex w plane. For any $z = x+iy$ in D, the real and imaginary parts of $w = f(z)$ are functions of x and y, which we denote $u(x,y)$, $v(x,y)$. Figure 3.9 depicts $w = f(z)$ as a mapping from region D of z plane to region

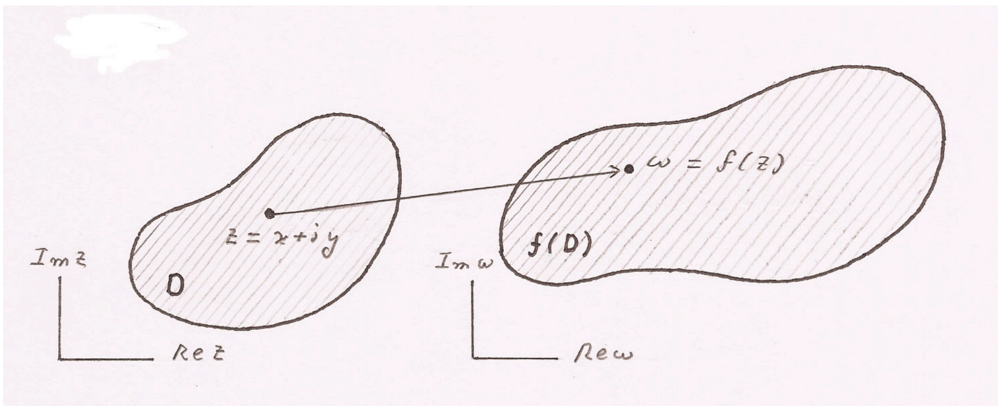

Figure 3.9

$f(D)$ of w plane.

As a simple example, consider the geometry of the mapping from $D : x = \operatorname{Re} z > 0$ into the w plane, given by $w = z^2$. In this case,

$$u = x^2 - y^2, \;\; v = 2xy. \tag{3.14}$$

For $x > 0$ fixed, (3.14) represents a parametric curve in the w plane, parametrized by y. These curves are of course level curves of x in the w plane. Eliminating y from (3.14), we obtain a relation between u and v parametrized by x,

$$u = x^2 - \frac{v^2}{4x^2}\,. \tag{3.15}$$

Geometrically, (3.15) represents a parabola opening in the $-u$ direction, and $w = 0$ is the focus of the parabola. As $x \to 0^+$, the parabola (3.15) becomes a "hairpin" wrapped around the negative u axis, as depicted in Figure 3.10. As x increases, starting from $x = 0^+$, the parabolas fill out the whole w plane except the negative u axis, so $f(D)$ is the whole w plane minus the negative u axis. An exercise treats the level curves of y, and other vector calculus details of the mapping $(x,y) \to (u(x,y), v(x,y))$. The complex function $w = z^2$ has

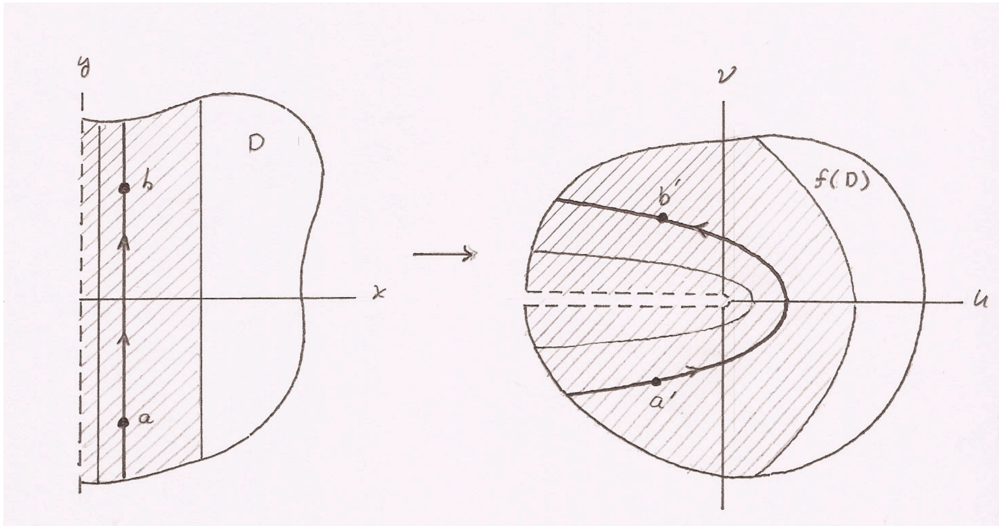

Figure 3.10

an electrostatic interpretation: The level curves of x in the u, v plane are level curves of the electric potential, due to a charged conductor along the $-u$ axis.

This agenda is to quickly establish the extensions of various essential functions from real variable calculus to complex variable calculus. The biggest prize of all for physicists is the extension of the real exponential function e^x to the complex exponential function e^z. A most "hands on" approach is via *complex power series*, which take the form

$$\sum_0^\infty a_n(z-a)^n, \tag{3.16}$$

where a and $a_0, a_1, \ldots$ are given complex constants. You can guess what the "hands on" method is: In the power series (1.48) of your favorite real function $f(x)$, simply replace x by z!

We mitigate this "smash and grab" with some preliminaries about the convergence of complex series. The infinite series

$$\sum_1^\infty a_k \tag{3.17}$$

of complex constants a_k *converges* if there is a complex number s so that

$$\lim_{n\to\infty} |a_1 + a_2 + \dots a_n - s| = 0.$$

That is, the modulus of the n-th partial sum of (3.17) minus s converges to zero as $n \to \infty$. As in the case of real series, we say that the complex series (3.17) *converges absolutely* if

$$\sum_1^\infty |a_k|$$

converges. The only difference from the previous definition of absolute convergence of real series is the meaning of $|a_k|$ as the *modulus* of complex numbers a_k. As in the case of real series, absolute convergence implies convergence. By use of inequalities $|\operatorname{Re} z| \le |z|$, $|\operatorname{Im} z| \le |z|$, it readily follows that the real and imaginary parts of $a_1 + a_2 \dots a_n$ converge to real and imaginary parts of s. So in practice, convergence of the complex series (3.17) is just convergence of its real and imaginary parts. Having reduced convergence of complex series to real series, we are "back to business as usual", and the convergence tests for real series come into play. For instance, $1 + \frac{i+1}{2} + \frac{(1+i)^2}{4} + \dots$ has $a_n = \left(\frac{1+i}{2}\right)^n$ so $|a_n| = \left(\frac{1}{\sqrt{2}}\right)^n$ and $\frac{|a_{n+1}|}{|a_n|} = \frac{1}{\sqrt{2}} < 1$, so the series is absolutely convergent by the ratio test, and hence convergent.

We examine the convergence of complex power series like (3.16). Let's start with complex geometric series. The algebra of the "telescoping sum trick" still applies because complex algebra is the same as real algebra. Hence,

$$s_n := 1 + z + \dots z^{n-1} = \frac{1 - z^n}{1 - z}$$

for $z \neq 1$. Observe that

$$\left| s_n - \frac{1}{1-z} \right| = \frac{|z|^n}{|1-z|} \to \begin{cases} 0 & \text{as } n \to \infty,\ |z| < 1, \\ \infty & \text{as } n \to \infty,\ |z| > 1. \end{cases}$$

Hence, we have

$$1 + z + z^2 + \dots = \frac{1}{1-z} \tag{3.18}$$

in $|z| < 1$, and divergence in $|z| > 1$. This phenomenon, of convergence inside a circle, and divergence outside, in general. If the complex power series is in powers of $z-a$, then the disk of convergence is centered about $z=a$. For instance, the ratio test applied to the moduli of terms in

$$\sum_1^\infty \frac{(z-1-i)^n}{n(\sqrt{2})^n}$$

indicates that the disk of convergence is

$$|z-1-i| < \sqrt{2}.$$

Convergence on the boundary circle can be investigated on a case-by-case basis. For instance, the geometric series

$$1 + z + z^2 + \dots$$

diverges on $|z|=1$, because the n-th term z^n does not converge to zero as $n \to \infty$. The series

$$1 + \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \dots \tag{3.19}$$

converges in $|z|<1$ and diverges in $|z|>1$, like the geometric series. But on $|z|=1$, the series of moduli, $1+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$, is convergent, so (3.19) converges on $|z|=1$.

The calculus of complex functions is scarcely begun. There is a whole theory of differentiation and integration of complex functions. The notion of real analytic functions (having all their derivatives in some interval) generalizes to complex analytic functions in regions D of the complex plane. The coefficients a_n in the power series (3.16) of an analytic function are expressed as in the Taylor series (1.48), but now the $f^{(n)}(a)$ are *complex* derivatives of $f(z)$ evaluated at some $z=a$ in the region D of analyticity, and so on. Here, a choice is made: To leave these riches for a later course, so as to have time to engage the complex exponential function, and its applications to ODE and PDE of mathematical physics. This alone is almost overwhelming.

The complex exponential function

denoted by e^z, is defined by simply substituting $z=x+iy$ in place of x in the real Taylor series $\sum_1^\infty \frac{x^n}{n!}$, so we have

$$e^z := \sum_0^\infty \frac{z^n}{n!}\,. \tag{3.20}$$

The ratio test establishes the convergence of this series for all z. We determine explicit formulas for real and imaginary parts of e^z as functions of x and y. For $y = 0$, (3.20) reduces to the Taylor series of the usual *real* exponential function e^x. The next natural step is to explore its values along the y axis. This is the famous Euler calculation:

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \dots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right), \end{aligned}$$

or

$$e^{iy} = \cos y + i \sin y. \tag{3.21}$$

If the exponentiation property

$$e^{z_1+z_2} = e^{z_1} e^{z_2} \tag{3.22}$$

is true for *complex* numbers z_1 and z_2, we'd have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y,$$

and we'd identify

$$\begin{aligned} u &:= \operatorname{Re} e^z = e^x \cos y, \\ v &:= \operatorname{Im} e^z = e^x \sin y. \end{aligned} \tag{3.23}$$

The proof of (3.22) can be pestered out of the series (3.20) and the binomial expansion:

$$\begin{aligned} e^{z_1} e^{z_2} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{m=0}^{N} \frac{N!}{m!(N-m)!} z_1^m z_2^{N-m} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} (z_1 + z_2)^N = e^{z_1+z_1}. \end{aligned}$$

The second equality is the same kind of rearrangement that was applied in the derivation of the two-variable Taylor series (2.17): Sum over m, n so $m + n = N$, and then sum over N. The third equality is the binomial expansion. In summary, the real and imaginary parts of the complex exponential are indeed given by (3.23).

Relatives of the exponential function in the complex plane

Replacing x by $z = x + iy$ in the real Taylor series for $\cos x$ and $\sin x$ gives the extensions of cosine and sine into the complex plane. For instance,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4} - \dots \tag{3.24}$$

and similarly for $\sin z$. An attempt to determine the real and imaginary parts of $\cos z$ by inserting $z = x + iy$ into (3.24) and applying the binomial expansion to $(x + iy)^n$ is awkward. Its much better to relate $\cos z$ and $\sin z$ to the complex exponential. If we redo the "Euler calculation" (3.21) with z replacing y, the algebra is *exactly* the same, leading to

$$e^{iz} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

or

$$e^{iz} = \cos z + i \sin z. \tag{3.25}$$

Replacing z by $-z$ in (3.25), and using the even and odd symmetry of $\cos z$ and $\sin z$, we have

$$e^{-iz} = \cos z - i \sin z. \tag{3.26}$$

We can solve (3.25), (3.26) for $\cos z$ and $\sin z$:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}),\ \sin z = \frac{1}{2!}(e^{iz} - e^{-iz}). \tag{3.27}$$

To find the real and imaginary parts of $\cos z$ as explicit functions of x and y, we rewrite the first of equations (3.27) as

$$\begin{aligned}\cos z &= \frac{1}{2}(e^{-y+ix} + e^{y-ix}) \\ &= \frac{1}{2}e^{-y}(\cos x + i \sin x) + \frac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \frac{1}{2}(e^{y} + e^{-y}) \cos x - \frac{i}{2}(e^{y} + e^{-y}) \sin x,\end{aligned}$$

or

$$\cos z = \cosh y \cos x - i \sinh y \sin x. \tag{3.28}$$

There is a similar calculation of the real and imaginary parts of $\sin z$.

The extension of the logarithm from the positive real axis to the complex plane is best done by "inversion": In the equation $w = e^z$, exchange the roles of w and z to get $z = e^w$, and we "solve" for $w = \log z$. Denoting the real and imaginary parts of $\log z$ by u and v, we find that the real and imaginary parts of the equation $z = e^w$ are

$$x = e^u \cos v, \;\; y = e^u \sin v.$$

We see that $u = \log r = \log |z|$, and v is one of the values of $\arg z$, so

$$\log z = \log |z| + i \arg z. \tag{3.29}$$

$\log z$ is "multivalued" because of the multivalued character of $\arg z$. (Recall the "helical ramp" graph of $\arg z$ in Figure 3.11.) The annoying multivaluedness goes away if we restrict the domain D of z's, so its simply connected and doesn't contain the origin. Figure 3.11 is an amusing choice for D: As you walk inside this "snail gut" from a to b, $\arg z$ continually increases, from say 0 at a to 4π at b.

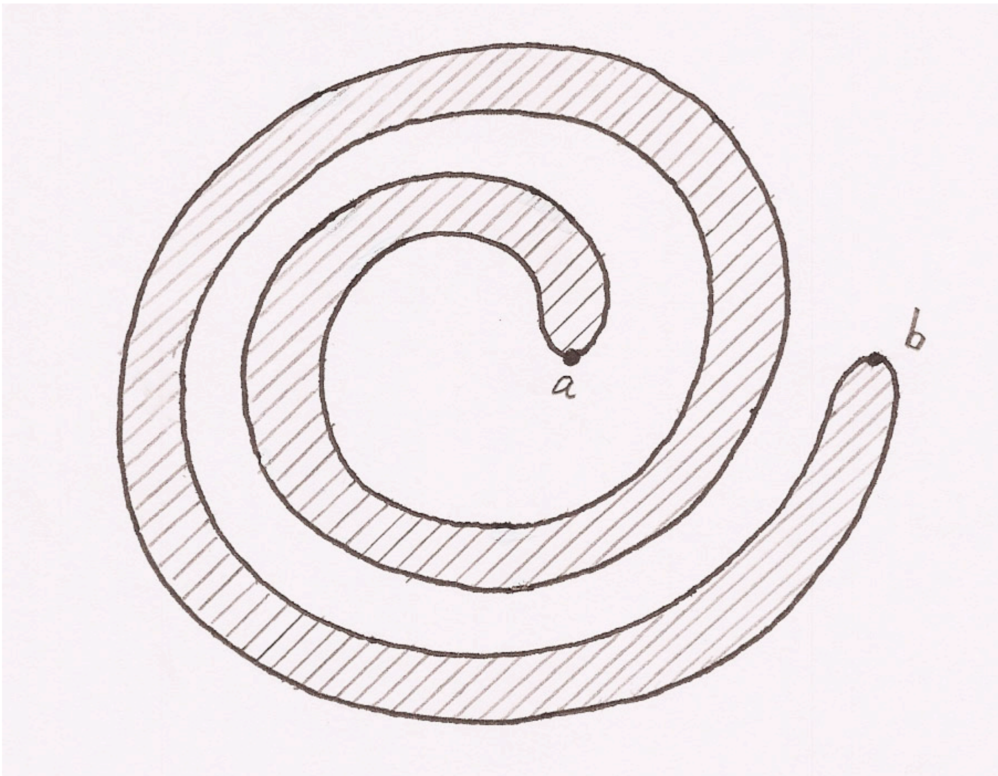

Figure 3.11

Basic calculus of the complex exponential

Consider the complex function $f(t)$ of real t, defined by

$$f(t) = e^{zt}, \tag{3.30}$$

where $z = x + iy$ is a complex constant. t-differentiation of $f(t)$ means differentiation of its real and imaginary parts,

$$\dot{f}(t) := (\operatorname{Re} f(t))^{\cdot} + i(\operatorname{Im} f(t))^{\cdot}.$$

From (3.23), we have

$$f(t) = e^{xt} \cos yt + ie^{xt} \sin yt,$$

so

$$\begin{aligned}\dot{f} &= xe^{xt}\cos yt - ye^{xt}\sin yt + i(xe^{xt}\sin yt + ye^{xt}\cos yt)\\ &= (x+iy)(e^{xt}\cos yt + ie^{xt}\sin yt) = ze^{zt},\end{aligned}$$

or

$$(e^{zt})^{\cdot} = ze^{zt}. \tag{3.31}$$

Integration of $f(t)$ means integration of real and imaginary parts according to

$$\int_a^b f(t)dt := \int_a^b \operatorname{Re} f(t)dt + i\int_a^b \operatorname{Im} f(t)dt. \tag{3.32}$$

From the definition and the fundamental theorem of calculus for real functions, it follows that

$$\int_a^b \dot{f}(t)dt = f(b) - f(a).$$

For $f(t) = e^{zt}$, we have (with the help of (3.31)),

$$\int_a^b e^{zt}dt = \frac{1}{z}(e^{zb} - e^{za}). \tag{3.33}$$

For z *real*, the differentiation and integration formulas (3.31), (3.33) are known from real variable calculus. The non-trivial new content is that they remain true for complex z.

There is another integral of extreme importance for physics: For real, positive a,

$$\int_{-\infty}^{\infty} e^{-at^2}dt = \sqrt{\frac{\pi}{a}}. \tag{3.34}$$

If a is complex with positive real part, (3.34) remains true provided we use the correct $\sqrt{a}$ in the right-hand side: We can represent a with positive real part by

$$a = re^{i\theta},$$

where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and the $\sqrt{a}$ that goes into (3.34) is

$$\sqrt{a} = \sqrt{r}e^{i\frac{\theta}{2}}.$$

Notice that $\arg\sqrt{a}$ lives in the sector $-\frac{\pi}{4} < \arg\sqrt{a} < \frac{\pi}{4}$. The proof of (3.34) is *much deeper* than the simple calculation of $\int_a^b e^{zt}dt$ in (3.33). It is based on *complex contour integration*. We give a highly simplified introduction which is sufficient to deal with (3.34).

Complex contour integration

Let $f(z)$ be a complex function represented by a convergent power series

$$f(z) = \sum_{0}^{\infty} a_n z^n \tag{3.35}$$

in some disk D centered about the origin. Next, let $C : z = z(t)$, $a \leq t \leq b$ be a parametric curve contained inside D. As t increases from a to b, $z(t)$ in the complex plane traces out the curve C, which "closes" because $z(b) = z(a)$. The *contour integral* of $f(z)$ over curve C is defined by

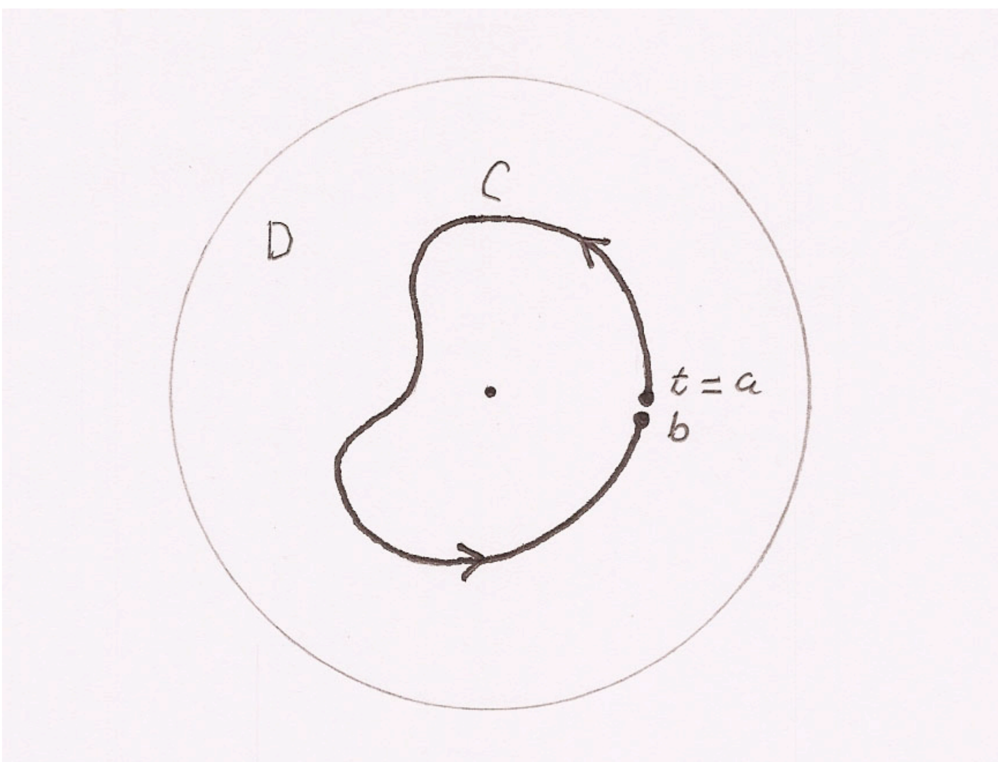

Figure 3.12

$$\int_C f(z)dz := \int_a^b f(z(t))\dot{z}(t)dt. \tag{3.36}$$

We substitute for $f(z)$ its power series (3.35) and formally interchange summation and integration:

$$\int_C f(z)dz = \sum_0^\infty a_n \int_a^b z^n(t)\dot{z}(t)dt. \tag{3.37}$$

In an exercise, you'll carry out elementary calculations which show that

$$\left(\frac{1}{n+1} z^{n+1}(t)\right)^{\cdot} = z^n(t)\dot{z}(t)$$

for any non-negative integer n. Hence,

$$\int_C f(z)dz = \sum_0^\infty \frac{a_n}{1+n}(z^{n+1}(b) - z^{n+1}(a))$$

which *vanishes* because $z(a) = z(b)$. Hence we have

$$\int_C f(z)dz = 0 \tag{3.38}$$

for any closed curve inside the disk D where the power series for $f(z)$ converges. This is a special case of the famous *Cauchy's theorem*. We've barely scratched the surface, but we've scratched it enough to demonstrate (3.34) for $\operatorname{Re} a > 0$.

The power series for $f(z) = e^{-z^2}$ converges for all z, and in this case C can be any closed curve in the complex plane. In particular, take C to be the "pie slice" in Figure 3.13. (Remember that $\sqrt{a}$ lies in the sector $-\frac{\pi}{4} < \arg\sqrt{a} < \frac{\pi}{4}$.) You can break the curve into three pieces: The line from 0 to R along the real axis, the circular arc, and the line segment from $Re^{i\arg\sqrt{a}}$ to 0. You can reparametrize each piece and compute

$$\int_C e^{-z^2}dz \tag{3.39}$$

as the sum of three integrals. For instance, the line segment from 0 to R is represented by $z = t$, $0 < t < R$ and its contribution to (3.39) is

$$\int_0^R e^{-t^2}dt. \tag{3.40}$$

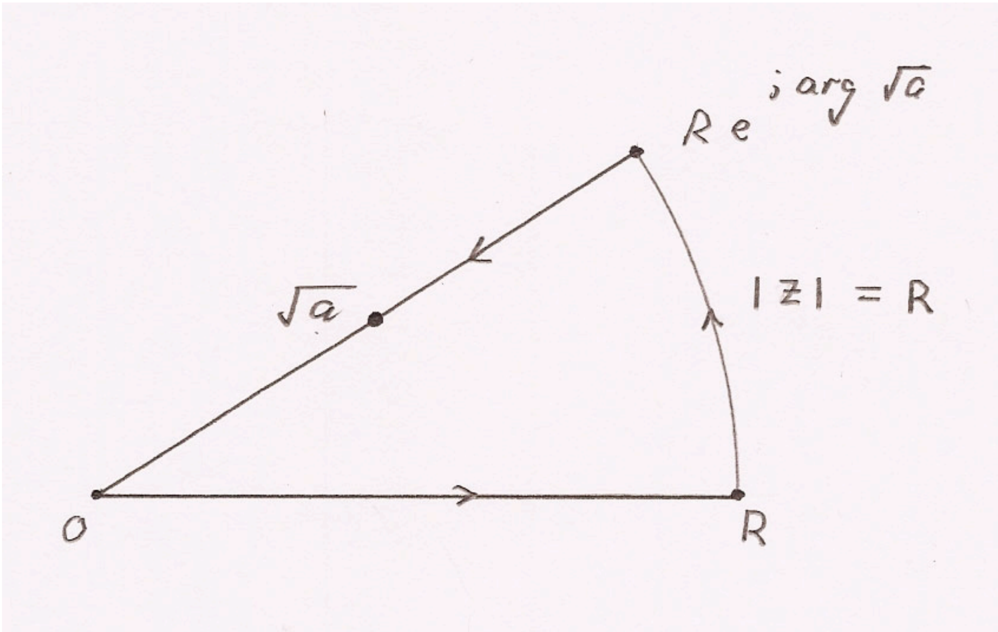

Figure 3.13

The line segment from $Re^{i\arg\sqrt{a}}$ to 0 can be represented by $z = \sqrt{a}t$, $0 < t < \frac{R}{\sqrt{|a|}}$ and its contribution to (3.39) is

$$\int_{\frac{R}{\sqrt{|a|}}}^{0} e^{-at^2}(\sqrt{a}dt) = -\sqrt{a}\int_0^{\frac{R}{\sqrt{|a|}}} e^{-at^2}dt. \tag{3.41}$$

The contribution from the circular arc vanishes as $R \to \infty$ because of the strong decay of $|e^{-z^2}|$ or $|z| \to\infty$ with $|\arg z| < \frac{\pi}{4}$. By Cauchy's theorem, (3.39) is *zero*, hence (3.40), (3.41) sum to zero in the limit $R \to \infty$, and we have

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-t^2}dt = \sqrt{a}\int_0^\infty e^{-at^2}dt,$$

so

$$\int_0^\infty e^{-at^2}dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

By evenness of the integrand, (3.34) follows. The limit $\mathrm{Re}\, a \to 0^+$ is interesting: Formally set $a = i$ in (3.39). We have $\sqrt{a} = e^{i\frac{\pi}{4}}$ and so

$$\int_{-\infty}^{\infty} e^{-it^2} dt = \sqrt{\pi} e^{-i\frac{\pi}{4}}$$

and the real and imaginary parts give *Fresnel integrals*,

$$\int_{-\infty}^{\infty} \cos t^2 dt = \int_{-\infty}^{\infty} \sin^2 t dt = \sqrt{\frac{\pi}{2}}.$$