Claim: \( \text{Pic}(\mathbb{P}^n) \cong \mathbb{Z} \)

Proof: Write \( \mathbb{P}^n \) as a union of open affine subsets \( U_i = \{ x_i \neq 0 \} \), \( 0 \leq i \leq n \). Here, the \( x_i \) refer to the standard coordinates on \( \mathbb{P}^n \). For an open subset \( U \) of \( \mathbb{P}^n \), the functions on \( U \) are ratios of homogeneous polynomials \( f/g \), where \( g \) does not vanish on \( U \) and \( \text{deg}(f) = \text{deg}(g) \). In particular, the ring of functions on \( U_i \) equals \( \mathbb{C}[\frac{x_i}{x_j}] \), \( j \neq i \).

Now, let \( E \) be a line bundle of \( \mathbb{P}^n \). \( E \) is trivializable on each of the \( U_i \), so let \( \alpha_i \in \Gamma(U_i, E) \) be a generator. For any open subset \( V \) of \( U_i \), the restriction of \( \alpha_i \) to \( V \) is a generator for \( \Gamma(V, E) \), and it should not cause confusion to call this element \( \alpha_i \) as well.

So, on \( U_0 \cap U_1 \), \( \alpha_0 \) and \( \alpha_1 \) are two generators of \( \Gamma(U_0 \cap U_1, E) \), and so we must have \( \alpha_1 = k \alpha_0 \), where \( k \) is an invertible element of \( \mathcal{O}(U_0 \cap U_1) = \mathbb{C}[\frac{x_0}{x_1}, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}] \). It is clear that the only invertible elements are of the form \( z(\frac{x_0}{x_1})^{d_1} \), where \( z \in \mathbb{C}^* \) and \( d_1 \in \mathbb{Z} \). Replacing \( \alpha_1 \) with \( \alpha_1/z \), we have \( \alpha_1 = (\frac{x_1}{x_0})^{d_1} \alpha_0 \).

Repeating this analysis, we must have \( \alpha_i = (\frac{x_i}{x_j})^{d_i} \alpha_0 \), for \( i = 1, 2, \ldots, n \), and integers \( d_i \in \mathbb{Z} \). It is clear that \( E \) is determined by the \( d_i \). Conversely, the \( d_i \) are determined by \( E \), because the only invertible elements ("gauge transformations") on each of the \( \mathcal{O}(U_i) \) are scalars. Our next step is to show that all \( d_i \) must be equal. Indeed, pick \( i \neq j \neq 0 \). By the same analysis as before we must have \( \alpha_i = z(\frac{x_i}{x_j})^m \alpha_j \) for some \( z \in \mathbb{C}^* \) and \( m \in \mathbb{Z} \). But then:

\[
\alpha_i = z \left( \frac{x_i}{x_j} \right)^m \alpha_j = z \left( \frac{x_i}{x_j} \right)^m \left( \frac{x_j}{x_0} \right)^{d_j} \alpha_0 = z \left( \frac{x_i}{x_j} \right)^m \left( \frac{x_j}{x_0} \right)^{d_j} \left( \frac{x_0}{x_i} \right)^{d_i} \alpha_i
\]

And so we must have \( z = 1 \) and \( m = d_j = d_i \). In fact, this shows that \( \alpha_i = (\frac{x_i}{x_j})^{d_i} \alpha_j \) for all \( i, j \), and for some fixed integer \( n \). Then, letting \( E \) correspond to \( d \), we have a bijection between \( \text{Pic}(\mathbb{P}^n) \) and \( \mathbb{Z} \). To see that this is a homomorphism: Observe that if \( E \) and \( E' \) are two line bundles that have generators \( \alpha_i \) of \( \Gamma(U_i, E) \) and associated integer \( d \) (resp. \( \alpha'_i \) of \( \Gamma(U_i, E') \), \( d' \)), then \( E \otimes E' \) have generators \( \alpha_i \otimes \alpha'_i \) on \( \Gamma(U_i, E \otimes E') \). Then on, say, \( U_0 \cap U_1 \) we have \( \alpha_1 \otimes \alpha'_1 = ((\frac{x_1}{x_0})^{d_1} \alpha_0) \otimes ((\frac{x_1}{x_0})^{d'_1} \alpha'_0) = (\frac{x_1}{x_0})^{d+d'} (\alpha_0 \otimes \alpha'_0) \).

\[ \square \]

Remark: There are exactly two ways of constructing an isomorphism of \( \text{Pic}(\mathbb{P}^n) \) with \( \mathbb{Z} \). The standard choice is to assign the integer \( d \) to the line bundle, denoted by \( \mathcal{O}_{\mathbb{P}^n}(d) \), where for \( U \subset \mathbb{P}^n \) open: \( \mathcal{O}_{\mathbb{P}^n}(d)(U) = \{ f/g \mid f, g \text{ homo.}, g \neq 0 \text{ on } U, \text{deg}(f) - \text{deg}(g) = d \} \). For \( E = \mathcal{O}_{\mathbb{P}^n}(1) \), a generator for \( \Gamma(U_0, E) \) is \( \alpha_0 = x_0 \), while a generator for \( \Gamma(U_1, E) \) is \( \alpha_1 = x_1 \). So \( \alpha_1 = (\frac{x_1}{x_0}) \alpha_0 \), and so the description in the proof above would also assign the number \( 1 \) to \( E \).