256B Algebraic Geometry Exercises

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Exercise 1.5: Any element of $\mathbb{C}[t, t^{-1}]$ can be written uniquely in the form $t^k f(t)$ where f is a polynomial with nonzero constant term. We call k the t-adic valuation of $t^k f(t)$.

We want to classify the double cosets

$$\mathrm{GL}_n(\mathbb{C}[t]) \backslash \mathrm{GL}_n(\mathbb{C}[t, t^{-1}]) / \mathrm{GL}_n(\mathbb{C}[t^{-1}]). \tag{1}$$

Observe that starting from a matrix in $\mathrm{GL}_n(\mathbb{C}[t, t^{-1}])$ and performing $\mathbb{C}[t]$-linear row operations resp. $\mathbb{C}[t^{-1}]$-linear column operations does not change the coset to which the matrix belongs.

Beginning from such a matrix, perform $\mathbb{C}[t]$-linear row operations as follows. First, consider the $\mathbb{C}[t]$-submodule of $\mathbb{C}[t, t^{-1}]$ generated by the elements of the first column. This submodule is t^k times an ideal of $\mathbb{C}[t]$ for some k, which is principal, hence it is generated by one element. Using row operations, replace one of the entries of the column by a generator, place it in the top row, and eliminate the rest of the entries in the column. Repeat for each column.

After performing the above row operations, we obtain an upper triangular matrix in the same coset as the original. Each of its diagonal entries is a unit in $\mathbb{C}[t, t^{-1}]$, hence must be an invertible constant times t^k for some $k \in \mathbb{Z}$ (and by multiplying the rows or columns by suitable constants we can assume WLOG that they have the form t^k).

Now perform the following sequence of row and column operations on each diagonal above the main diagonal (do not move on to a new diagonal until each entry in the current diagonal is zero). Using row and column operations, ensure that each term of the diagonal entry $p(t)$ has the property that if t^ℓ is the main diagonal entry to the left of it and t^d is the main diagonal entry below it, then $p(t)$ contains no terms of degree greater than or equal to m or less than or equal to n. We will abbreviate this situation using the 2×2 matrix

$$\begin{bmatrix} t^\ell & p(t) \\ 0 & t^d \end{bmatrix} \tag{2}$$

where $p(t) = \sum_{\ell+1}^{d-1} p_n t^n$. Let $k < 0$ be the negative integer such that the greatest power of t dividing $t^k p(t)$ is t^ℓ, and perform the column operation

$$\begin{bmatrix} t^\ell - t^k p(t) & p(t) \\ -t^{d+k} & t^d \end{bmatrix}. \tag{3}$$

Now perform row operations to eliminate the bottom left entry. The result will have the form

$$\begin{bmatrix} t^{\ell'} & p'(t) \\ 0 & t^{d'} \end{bmatrix} \tag{4}$$

where ℓ' is the minimum of the t-adic valuation of $t^\ell - t^k p(t)$ (which is greater than ℓ) and $d+k$. But since $p(t)$ by assumption contains no terms of degree greater than or equal to m, $d+k$ is necessarily also greater than ℓ. Hence our row and column operations have ensured that $\ell' > \ell$ and $d' < d$. If $p'(t) = 0$, we are done and can move on to a new entry in the same diagonal or to a new diagonal. Otherwise, we repeat. After a finite number of steps, we will have $\ell^{(n)} \geq d^{(n)}$, at which point $p^{(n)}(t)$ necessarily vanishes and we can move on to a new entry or a new diagonal.

The above algorithm eventually produces a diagonal entry with entries of the form t^k for some k, which shows that every double coset in

$$\mathrm{GL}_n(\mathbb{C}[t]) \backslash \mathrm{GL}_n(\mathbb{C}[t, t^{-1}]) / \mathrm{GL}_n(\mathbb{C}[t^{-1}]). \tag{5}$$

contains a matrix of the desired form. It remains to be shown that this matrix is unique up to permutation of its diagonal entries. So suppose two diagonal matrices D, D' with diagonal entries $t^{k_i}, t^{k'_i}$ lie in the same double coset, hence there exist $A \in \mathrm{GL}_n(\mathbb{C}[t])$ and $B \in \mathrm{GL}_n(\mathbb{C}[t^{-1}])$ such that

$$D' = A^{-1} D B. \tag{6}$$

By taking determinants we see that $\sum k_i = \sum k'_i$. Rewrite the above identity as

$$AD' = DB. \tag{7}$$

Let a_{ij}, b_{ij} be the entries of A, B. Then the above gives

$$t^{k_j} a_{ij} = t^{k'_i} b_{ij} \tag{8}$$

or

$$b_{ij} = t^{k_j - k'_i} a_{ij}. \tag{9}$$

Since $\sum k_i = \sum k'_i$, it follows that $\sum (k_i - k'_i) = 0$. If $\sigma \in S_n$ is a permutation, it follows that

$$\sum_i (k_{\sigma(i)} - k'_i) = 0 \tag{10}$$

hence that either $k_{\sigma(i)} = k'_i$ for all i (in which case we are done) or that there exists an i such that $k_{\sigma(i)} - k'_i > 0$. But $a_{ij} \in \mathbb{C}[t]$ and $b_{ij} \in \mathbb{C}[t^{-1}]$, so this is possible if and only if $a_{ij} = b_{ij} = 0$. Since A, B are not identically zero, there must exist a permutation σ such that $k_{\sigma(i)} = k'_i$ for all i, and the conclusion follows.