Mid-term 2 Review 10. 26.

§4.1 vector spaces & Subspaces.

- Axiom of each (conceptually understand) subspaces are vecterspaces.

Example.

- vecterspaces 1.) $\mathbb{R}^n$, $n \geq 0$
  2.) $S = \mathbb{R}^\infty = \{(a_1 \cdots)\}$
  3.) $S = \{(a_1 \cdots)$ eventually $0\}$
  4.) functions $\{f; \mathbb{R} \to \mathbb{R}\}$
  5.) continous function $\{f: \mathbb{R} \to \mathbb{R}$ continous$\}$
  6.) polynomial $\{f: \mathbb{R} \to \mathbb{R}\}$
  7.) $P_{\leq n}$ $\{f: \mathbb{R} \to \mathbb{R}$ polynomial of degree $\leq n\}$
  ⋮ ⋮ ⋮

- Subspaces 1.) span $\{v_1 \cdots v_k\} \subset V$
  2.) Null$(T) \subset V$, $T: V \to W$
  "cut out of linear equation"
  "linear combination of some vectors"
  3.) Image$(T) \subset W$, $T: V \to W$

Special case: $V = \mathbb{R}^n$, $W = \mathbb{R}^n$
T given by A ~ Null(A)
Image(A) = col(A)

Axioms of subspaces $H \subset V$

1.) $0 \in H$
2.) $\bar{u}, \bar{v} \in H$, then $\bar{u} + \bar{v} \in H$
3.) $\bar{u} \in H$, then $c\bar{u} \in H$

Non-Examples

$A\bar{x} = \bar{b}$ for $\bar{b} \neq \bar{0}$ solns set $\begin{cases} \emptyset \\ H + V \end{cases}$

H: Null(A), V: a soln.

is not a subspace.

move # away from origin.

§4.2 Null spaces, column space & linear transformations

Def.

Axioms of linear transformations.

Ex. $m \times n$ matrix define lin. Transf. $T: \mathbb{R}^n \to \mathbb{R}^k$ by acting on vectors

Comfortable working with abstract V, W & abstract T

Ex. $T: P_{\le 3} \to P_{\le 4}$

$$T(P(x)) = xP(x) + P'(2)$$

§4.3 Linearly independent set & basis

Def.

$v_1 \dots v_n$ lin. indep $\in \mathbb{R}^n$	$v_1 \dots v_n$ span $\mathbb{R}^n$
$k \le n$	$k \ge n$.
Removing vectors	adding vectors.
$n\{\underbrace{\begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_n \\ \vert & & \vert \end{bmatrix}}_{k}$ Pivot in each col.	$\begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_n \\ \vert & & \vert \end{bmatrix}$ Pivot in each row
$\Uparrow$ injective. $Null(A) = \{0\}$.	$\Uparrow$ surjective. $Image(A) = \mathbb{R}^n$

Basis. $\Rightarrow$ linearly independent + span

$\Rightarrow$ just Right size ☺

useful construction.

1.) $v_1 \cdots v_k$ is lin. indep. in $\mathbb{R}^n$, can always find $v_{k+1} \cdots v_n$ s.t. $v_1 \cdots v_n$ is a basis.

2.) if $v_1 \cdots v_k$ span $\mathbb{R}^n$, then we can always Remove vectors s.that remaining are a basis $\begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_k \\ \vert & & \vert \end{bmatrix}$

Remember to go back & find original basis

3.) $v_1 \cdots v_k$ is lin dep.

first free col. of $[v_1 \cdots v_k]$ $\Updownarrow$

some $v_i \in \text{span}\{v_1 \cdots v_{i-1}\}$.

- single vector v is lin. indep: $v \neq 0$
- span $\phi = \{0\}$

§4.5 & 4.6. Dimension & Rank.

Dimension: Any two basis of V have the same # of vectors

$\Rightarrow \dim(V) = \#$ of basis.

Ex. Null $\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 3 \end{bmatrix} \subset \mathbb{R}^4$.

$\dim = 2$ ← free columns.

- dim Null; dim Image = dim col.

free pivot column pivot columns

Personal Note could be ignored ☺ → [P.S. Basis for Null ⇒ after
Basis for Image ⇒ origin.]

$\dim H \leq \dim V$ if $H \subset V$, subspaces.

$H < T = \dim(\text{Image}(T))$

Rank Theorem.

Suppose $\dim V =$ finite. $T: V \to W$ lin. transf

$\dim V = \dim \text{Null}(T) + \text{rank}(T)$

For Matrices, $m \times n$

$n = \#$ free var. + # of pivot colns.

Nullity $T = \dim \text{Null}(T)$.

Relationship of rh and Nullity to matrix alg.

Ex. A, B $m \times n$, $\text{rk}A = n$, $\text{rk}B = b$. $\text{rk}(A+B) =$ what Possibly

$[0, a+b]$ or say $[0, \min(a+b, n)]$

$A_{(m\times n)}$ $B_{(n\times k)}$

$rk(A)=a.$ $rk(B)=b.$

$0 \le rank(AB) \le \min\{a,b\}.$

§4.4 coordinate.

Given a basis $\beta\{v_1 \cdots v_n\}$. can $T_\beta : V \to \mathbb{R}^n$ (invertible).

$T_\beta(v) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ so $V = a_1v_1 + \cdots a_nv_n$

$T_\beta^{-1} : \mathbb{R}^n \to V$ $\quad T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1v_1 + \cdots + a_nv_n.$

$[v_1' \cdots v_n' : \bar{v}]$ ← solving lin. syst. multiplying & add.

Should be able to calculate T_β, T_β^{-1} in examples.

Find matrix of $T: V \to W$ given basis β of V, $\dim \le n$, $H \subset \mathbb{R}^n$

Note: $P_\beta = T_\beta^{-1}$ $\quad P_\beta^{-1} = T_\beta$

α of W.

$A = [T]_\beta^\alpha = [[T(v_1)]_\alpha \cdots]$ Null Image.

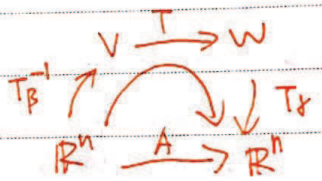
A also $= T_\gamma T T_\beta^{-1}$

$T_\beta : \mathbb{R}^n \xrightarrow{\varepsilon} \mathbb{R}^n \xrightarrow{P}$

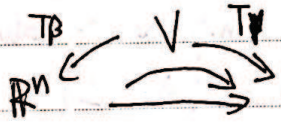
$= P_{\beta\leftarrow\varepsilon}$

$= P_{\varepsilon\leftarrow\beta}^{-1}$

$= P_\beta^{-1}$

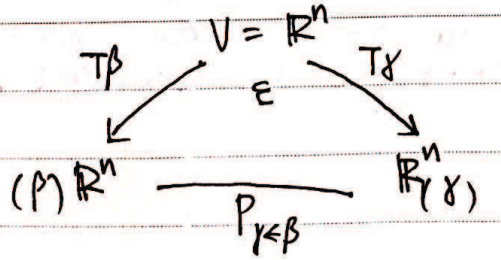


§4.7. change of Basis.



$P_{\gamma\leftarrow\beta} = T_\gamma T_\beta^{-1} = \left[[v_1']_\gamma, \ldots [v_n']_\gamma\right]$

$[v]_\gamma = P_{\gamma\leftarrow\beta}\,[v]_\beta$



$P_{\gamma\leftarrow\beta} = P_{\gamma\leftarrow\varepsilon} \cdot P_{\varepsilon\leftarrow\beta} = T_\gamma T_\beta^{-1}$

$= P_\gamma^{-1} \cdot P_\beta$

computed by $\begin{matrix} P_\beta = [v_1' \cdots v_n'] \\ P_\gamma^{-1} \end{matrix}$ or $[P_\gamma : P_\beta]$

$\sim [I_n : P_\gamma^{-1}P_\beta]$

§5.1 E-value & E-vector.
§5.2

$A\bar{x} = \lambda\bar{x} \Rightarrow (A-\lambda I)x = 0 \quad (x \neq 0)$

$\Rightarrow$ Null $(A-\lambda I)$ that $\neq 0$

$\Leftrightarrow \det(A-\lambda I) = 0.$

$\Rightarrow \lambda$ is root of $\chi_A(t) = \det(A-\lambda I)$

Algorithm.

Step 1. $\chi_A(t)$ { real soln. Sometimes can't be factorized.
complex soln. always can be factored

Step 2. Find basis of eigenspaces.

$E_\lambda = \text{Null}(A-\lambda I)$ $\quad 1 \leq \dim E_\lambda \leq$ mult. of λ

§5.3 diagonalization.

if $D = P^{-1}AP$, $\Rightarrow$ A is diagonalizable.

↑ diagonalize matrix ↑ Basis of e-vectors.

D is not unique

§5.4 Similarity.

$A_{(n\times n)} \sim B_{(n\times n)} \Rightarrow A = P^{-1}BP$ "A is B in alternative basis β"

$\Rightarrow A = P^{-1}BP_{\varepsilon\leftarrow\beta} \overset{\text{Same as}}{=\!=} [A[\bar{v}]_\beta]_\varepsilon = B\bar{v}$

§5.5

Suppose 2x2 matrix A has complex e-values $\lambda, \bar{\lambda}$

should be able to find Null $(A-\lambda I)$

Then diagonalize. $D = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$

if $\lambda_+ = a+ib$. $\quad A \sim \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \frac{a}{r} & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

↑ Stretch ↑ rotation.

Shears: $\begin{bmatrix} a & b\neq 0 \\ 0 & a \end{bmatrix}$ No Hope of diagonalizable.

§ Question time

$$\text{Null}(A) = \text{Row}(A)^{\perp}$$

$$\begin{bmatrix} -a_1- \\ -a_2- \\ \vdots \\ -a_n- \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \cdot x \\ \vdots \\ a_n \cdot x \end{bmatrix}$$

Ex. $H = \text{Span}\left\{ v_1 = \begin{bmatrix} 7 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 6 \end{bmatrix} \right\}$. Find a Basis of $H^{\perp}$

$$A = \begin{bmatrix} -v_1- \\ -v_2- \end{bmatrix}$$

$H = \text{Row}(A)$.

Then Apply $\text{Null}(A) = \text{Row}(A)^{\perp}$

$\Rightarrow$ Find Null(A) and then find basis.