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Cyclic homology and equivariant homology

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Introduction

The purpose of this paper is to explore the relationship between the cyclic homology and cohomology theories of Connes [9–11], see also Loday and Quillen [20], and $\mathbb{T}$ equivariant homology and cohomology theories. Here $\mathbb{T}$ is the circle group. The most general results involve the definitions of the cyclic homology of cyclic chain complexes and the notions of cyclic and cocyclic spaces so precise statements will be postponed until § 3. In this introduction we explain some of the formal similarities between the cyclic theory and the equivariant theory and give two examples where the general results apply.

Let A be an associative algebra over a commutative ring K. Then one can form the cyclic homology $HC_*(A)$ and cohomology $HC^*(A)$ of A. These groups have periodicity operators

$$HC_n(A) \to HC_{n-2}(A), \qquad HC^n(A) \to HC^{n+2}(A).$$

Connes [10] has defined products in cyclic cohomology and using this product structure $HC^*(K)$ becomes a polynomial ring $K[u]$ where u has degree 2. The groups $HC^*(A)$ now become modules over this polynomial ring and the action of u corresponds to the periodicity operator. Thus it seems reasonable to regard $K[u]$ as the natural coefficients for cyclic cohomology and then to make $HC_*(A)$ into a module over $K[u]$ by using the periodicity operator. However it is clear that every element of $HC_*(A)$ is u-torsion so $HC^*(A)$ and $HC_*(A)$ cannot be dual over $K[u]$.

In § 2 we introduce a variant of cyclic homology $HC_*^-(A)$ which should be thought of as dual, over $K[u]$, to $HC^*(A)$. In particular $HC_*^-(A)$ is a module over $K[u]$ where the action of u corresponds to a periodicity operator

$$HC_n^-(A) \to HC_{n-2}^-(A).$$

The algebraic properties of $HC_*^-(A)$ will be studied in [17] where the statement that $HC_*^-(A)$ is dual, over $K[u]$, to $HC^*(A)$ is justified. It is shown in [17] that one can define products in $HC_*^-(A)$ so that $HC_*^-(K)$ and $K[u]$ become isomorphic rings, where here u has degree -2, and the module structure of $HC_*^-(A)$ over $HC_*^-(K)$ is the same as the module structure defined by the periodicity operator. One feature of $HC_*^-(A)$ is that it can be non-zero in all degrees, positive or negative.

Next one can invert the action of u on $HC_*^-(A)$ and form the groups

$$H\hat{C}_*(A) = u^{-1} HC_*^-(A).$$

These groups become modules over $K[u, u^{-1}]$. These various forms of cyclic homology are related by a long exact sequence

$$\ldots \to HC_n^-(A) \to H\hat{C}_n(A) \to HC_{n-2}(A) \to HC_{n-1}^-(A) \to \ldots.$$

There is a clear and precise analogy with the Tate homology of groups.

There is a similar formal structure in $\mathbb{T}$ equivariant homology and cohomology theories, [1, 24], which we now outline. Let Z be a space with a circle action, then the $\mathbb{T}$ equivariant cohomology of Z is defined to be

$$H_{\mathbb{T}}^*(Z) = H^*(E\mathbb{T} \times_{\mathbb{T}} Z)$$

where $E\mathbb{T}$ is a contractible space on which $\mathbb{T}$ acts freely and the coefficients are taken in the ring K. These equivariant cohomology groups are modules over $H_{\mathbb{T}}^*(\text{point}) = H^*(B\mathbb{T})$ which will be identified with $K[u]$. One can now form the localised equivariant cohomology theory $\hat{H}_{\mathbb{T}}^*(Z) = u^{-1} H_{\mathbb{T}}^*(Z)$. There is a third equivariant cohomology theory, which will be denoted $G_{\mathbb{T}}^*(Z)$, related to $H_{\mathbb{T}}^*(Z)$ and $\hat{H}_{\mathbb{T}}^*(Z)$ by a long exact sequence

$$\ldots \to H_{\mathbb{T}}^n(Z) \to \hat{H}_{\mathbb{T}}^n(Z) \to G_{\mathbb{T}}^{n+2}(Z) \to H_{\mathbb{T}}^{n+1}(Z) \to \ldots.$$

There are analogous equivariant homology theories related by the exact sequence

$$\ldots \to G_n^{\mathbb{T}}(Z) \to \hat{H}_n^{\mathbb{T}}(Z) \to H_{n-2}^{\mathbb{T}}(Z) \to G_{n-1}^{\mathbb{T}}(Z) \to \ldots.$$

Here one should think of $G_*^{\mathbb{T}}(Z)$ as dual to $H_{\mathbb{T}}^*(Z)$ and $H_*^{\mathbb{T}}(Z)$ as dual to $G_{\mathbb{T}}^*(Z)$ over $K[u]$. We will refer to the three above long exact sequences as the fundamental long exact sequences of the appropriate theories.

We now give two results, special cases of theorems in § 3, which illustrate the relation between the cyclic theories and the equivariant theories. For both these results we need to extend the definitions of cyclic homology so that $HC_*^-(A)$, $H\hat{C}_*(A)$ and $HC_*(A)$ are all defined for differential graded algebras A. This is straightforward, see § 3.

Let X be a topological space and let $L(X) = \mathrm{Map}(\mathbb{T}, X)$ be the space of free loops in X. Then the circle acts on $L(X)$ by rotating loops. Let $S^*(X)$ be the singular cochain complex of X made into an associative differential graded algebra using the Alexander-Whitney product [16, Chap. 29, p. 193]. Grade $S^*(X)$ negatively so that the differential decreases degree by one; this means that $HC_*(S^* X)$ will, in general, be non-zero in all degrees.

Theorem A. *If X is simply connected, there are natural isomorphisms*

$$HC^{-}_{-n}(S^*X) \cong H^n_{\mathbb{T}}(LX) \quad \text{as } K[u] \text{ modules}$$
$$H\hat{C}_{-n}(S^*X) \cong H^n_{\mathbb{T}}(LX) \quad \text{as } K[u,u^{-1}] \text{ modules}$$
$$HC_{-n}(S^*X) \cong G^n_{\mathbb{T}}(LX) \quad \text{as } K[u] \text{ modules.}$$

These isomorphisms throw the fundamental exact sequence of cyclic homology onto the one for equivariant cohomology.

In this result it is probably more natural, particularly as we have graded cochains negatively, to grade cohomology negatively and to change the grading of $\hat{H}^*_{\mathbb{T}}$ and $G^*_{\mathbb{T}}$ so that the result becomes $HC^-_n(S^*X) \cong H^{-n}_{\mathbb{T}}(LX)$ and so on. For the moment however we will stick to the usual grading conventions for cohomology.

Corollary. *Let X be simply connected compact manifold. Take the coefficients K to be $\mathbb{R}$ or $\mathbb{C}$ and let Ω^*X be the (differential graded) algebra of (real or complex as appropriate) differential forms on X. Then there are natural isomorphisms*

$$HC^{-}_{-n}(\Omega^*X) \cong H^n_{\mathbb{T}}(LX) \quad \text{as } K[u] \text{ modules}$$
$$H\hat{C}_{-n}(\Omega^*X) \cong \hat{H}^n_{\mathbb{T}}(LX) \quad \text{as } K[u,u^{-1}] \text{ modules}$$
$$HC_{-n}(\Omega^*X) \cong G^n_{\mathbb{T}}(LX) \quad \text{as } K[u] \text{ modules.}$$

These isomorphisms throw the fundamental exact sequence of cyclic homology onto the one for equivariant cohomology.

The original motivation for this work was to compute the cyclic homology of Ω^*X.

Now let G be a topological group and let BG be its classifying space. Let $S_*(G)$ be the singular chain complex of G made into an associative differential graded algebra using the Eilenberg-McLane shuffle product [16, Chap. 29, 29.27] $S_*(G) \otimes S_*(G) \to S_*(G \times G)$ and the map induced by the product law $G \times G \to G$.

Theorem B. *Let G be a topological group, then there are natural isomorphisms*

$$HC^-_n(S_*G) \cong G^{\mathbb{T}}_n(LBG) \quad \text{as } K[u] \text{ modules}$$
$$H\hat{C}_n(S_*G) \cong \hat{H}^{\mathbb{T}}_n(LBG) \quad \text{as } K[u,u^{-1}] \text{ modules}$$
$$HC_n(S_*G) \cong H^{\mathbb{T}}_n(LBG) \quad \text{as } K[u] \text{ modules.}$$

These isomorphisms throw the fundamental exact sequence of cyclic homology onto the one for equivariant homology.

These two results are important motivation for the rest of this paper, they were discovered independently but they are clearly related. Theorem B is a version of a theorem due to Goodwillie [14]; it is also proved by Burghelea and Fiedorowicz [7]. The approach used here is rather different from that of [14] and [7]. One of the aims has been to give a unified treatment of these two results, indeed from the point of view adopted in this paper these two theorems, or more accurately the general results 3.1 and 3.2, appear as two sides of the same coin. The main new ingredient we introduce is the study of cocyclic spaces, which leads to Theorem A. Two other features of the approach used here are the systematic use of the three theories $HC_*^-(A)$, $H\hat{C}_*(A)$ and $HC_*(A)$, and the deduction of the main results from the geometrical interpretation of Connes' B operator (see § 4).

There are cyclic cohomology versions of both these theorems. The simplest statements are $HC^n(S^*X)\cong G_n^{\mathbb{T}}(LX)$ and $HC^n(S_*G)\cong H_{\mathbb{T}}^n(LBG)$ but we leave the reader to formulate the results precisely and to trace the proofs from the arguments we give.

The rest of this paper is organised as follows; § 1 and § 2 are preliminary sections which contain a discussion of Connes category Λ, cyclic objects and the definitions, and some elementary properties, of the various cyclic homology groups. Much of the contents of these two sections can be found in the literature [8–11, 18–20]; we have included this material in an effort to be self contained. In § 3 we state the main results in full detail and begin their proofs. The proofs are completed in § 4 and § 5. In § 6 we give the proofs of Theorems A and B and in § 7 we discuss an application of Theorem A.

§ 1. Cyclic objects – definitions and examples

We begin by describing Connes' category Λ, [11]. Start with the category Δ whose objects are the finite ordered sets $\mathbf{n}=\{0,1,\dots,n\}$, and whose morphisms $\Delta(\mathbf{n},\mathbf{m})$ are the order preserving maps $s\colon \mathbf{n}\to\mathbf{m}$. Here order preserving means that if $i\leqq j$ then $s(i)\leqq s(j)$. Connes extends Δ by introducing cyclic permutations of $\mathbf{n}$. In detail the objects of Λ are the sets $\mathbf{n}$ and the morphisms are $\Lambda(\mathbf{n},\mathbf{m}) =\Delta(\mathbf{n},\mathbf{m})\times K(\mathbf{n})$ where $K(\mathbf{n})$ is the group of cyclic permutations of $\mathbf{n}$. To define the composition law start with $t\in\Delta(\mathbf{n},\mathbf{m})$ and $u\in K(\mathbf{m})$ and construct new elements $t^*u\in K(\mathbf{n})$ and $u_*(t)\in\Delta(\mathbf{n},\mathbf{m})$ as follows. Let A_i be the set $t^{-1}i\subset\mathbf{n}$; A_i is given the ordering it inherits as a subset of $\mathbf{n}$. Now define B_i by $B_{u(k)}=A_k$; again B_i is ordered. We will always give an ordered disjoint union of ordered sets its natural ordering. As sets $\mathbf{n}$ is the same as $B_0\cup\dots\cup B_m$ but not necessarily as ordered sets; we get a new ordering of $\mathbf{n}$. Write t^*u for the permutation

$$(i+1)\text{-th element in the new ordering}\to i.$$

It is easy to check that if u is cyclic then so is t^*u. Now define u_*t to be $ut(t^*u)^{-1}$; it is easy to check that it is order preserving. Define the composition law by the formula

$$(s,u)(t,v)=(s\cdot(u_*t),(t^*u)\cdot v).$$

The morphisms of Λ are generated (using composition) by:

(a) The face maps $\delta_i\in\Lambda(\mathbf{n-1},\mathbf{n})$, $0\leqq i\leqq n$; the unique injective order preserving map whose image does not contain i.

(b) The degeneracy maps $\sigma_i\in\Lambda(\mathbf{n+1},\mathbf{n})$, $0\leqq i\leqq n$; the unique surjective order preserving map which repeats i.

(c) The cyclic permutation $\tau_n\in\Lambda(\mathbf{n},\mathbf{n})$; $\tau_n(i)=i-1 \bmod n+1$.

These generators satisfy the usual cosimplicial relations together with extra "cyclic relations". These relations are easy to work out; for the convenience of the reader we list them below:

1.1 (a) $\delta_j\delta_i = \delta_i\delta_{j-1}$ $\quad i<j$

(b) $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$ $\quad i\leqq j$

(c) $\sigma_j\delta_i = \delta_i\sigma_{j-1}$ $\quad i<j$

$= 1$ $\quad i=j$ or $i=j+1$

$= \delta_{i-1}\sigma_j$ $\quad i>j+1$

(d) $\tau_n\delta_i = \delta_{i-1}\tau_{n-1}$ $\quad 1\leqq i\leqq n$

$\tau_n\delta_0 = \delta_n$

(e) $\tau_n\sigma_i = \sigma_{i-1}\tau_{n+1}$ $\quad 1\leqq i\leqq n$

$\tau_n\sigma_0 = \sigma_n\tau_{n+1}^2$

(f) $\tau_n^{n+1} = 1$.

This category Λ is self dual, that is there is an equivalence between Λ and its opposite category. This equivalence is given by the identity on objects and on morphisms by $s \to s^*$ where

$$\begin{aligned}
&\delta_i^* = \sigma_i && \delta_i\colon \mathbf{n-1} \to \mathbf{n} \quad 0\leqq i\leqq n-1\\
&\delta_n^* = \sigma_0\tau_n^{-1} && \delta_n\colon \mathbf{n-1} \to \mathbf{n}\\
&\sigma_i^* = \delta_{i+1} &&\\
&\tau_n^* = \tau_n^{-1}. &&
\end{aligned}$$

It is easy to check that if s is in $\Lambda(\mathbf{n}, \mathbf{m})$ then $s^{**} = \tau_m^{-1} s \tau_n$.

Following Connes [11] we define a cyclic object in a category $\mathbf{C}$ to be a contravariant functor $\Lambda \to \mathbf{C}$ and a cocyclic object in $\mathbf{C}$ to be a covariant functor $\Lambda \to \mathbf{C}$. Cyclic (or cocyclic) objects in $\mathbf{C}$ form a category with morphisms natural transformations of functors. Using the equivalence of Λ with its opposite each cyclic object F gives rise to a cocyclic object F^0, and vice versa. Given a cyclic object in $\mathbf{C}$ we can always regard it as a simplicial object in $\mathbf{C}$, that is a contravariant functor from $\Delta \to \mathbf{C}$, simply by forgetting the morphisms τ_i; similarily we can regard a cocyclic object as a cosimplicial object.

Example 1.2. Let **Top** be the category of topological spaces and continuous maps. Let X be a space and let $\mathbf{X}\colon \Lambda \to \mathbf{Top}$ be the cyclic space

$$\mathbf{X}(\mathbf{n}) = \mathrm{Map}(\mathbf{n}, X) = X^{n+1}.$$

We get a cocyclic space $\mathbf{X}^0$. This cocyclic space arises in the proof of Theorem A so we describe its structure in detail. The maps δ_i, σ_i, and τ_n induce the following maps of spaces

$$\begin{aligned}
\delta_i(x_0, \ldots, x_{n-1}) &= (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_{n-1}), \qquad 0\leqq i\leqq n-1\\
\delta_n(x_0, \ldots, x_{n-1}) &= (x_0, x_1, \ldots, x_{n-1}, x_0)\\
\sigma_i(x_0, \ldots, x_{n+1}) &= (x_0, \ldots, x_i, x_{i+2}, \ldots, x_{n-1})\\
\tau_n(x_0, \ldots, x_n) &= (x_1, \ldots, x_n, x_0).
\end{aligned}$$

Example 1.3. Let G be an associative topological monoid with unit e, then make $\mathbf{n} \to G^{n+1}$ into a cyclic space as follows: Write d_i, s_i and t_n for the maps induced by δ_i, σ_i, and τ_n, then

$$d_i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n)$$
$$s_i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, e, x_{i+1}, x_{i+2}, \ldots, x_n)$$
$$t_n(x_0, \ldots, x_n) = (x_n, x_0, x_1, \ldots, x_{n-1}).$$

Write $\mathbf{K}_*$ for the category of chain complexes over K. If C is a chain complex over K and $x \in C$ then we write $|x|$ for the degree of x. If C and D are chain complexes then we use the usual sign convention for the differential in $C \otimes D$, $d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$.

Example 1.4. Let A be a DGA, that is an associative differential graded algebra, over K. Then make $\mathbf{n} \to A^{\otimes n+1}$ into a cocyclic chain complex $\mathbf{A}$: $\Lambda \to \mathbf{K}_*$ as follows

$$\delta_i(a_0 \otimes \ldots \otimes a_{n-1}) = a_0 \otimes \ldots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \ldots \otimes a_{n-1}$$
$$\sigma_i(a_0 \otimes \ldots \otimes a_{n+1}) = a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n$$
$$\tau_n(a_0 \otimes \ldots \otimes a_n) = (-1)^r (a_1 \otimes \ldots \otimes a_n \otimes a_0)$$

where $r = |a_0|(|a_1| + \ldots + |a_n|)$.

We now have the cyclic chain complex $\mathbf{A}^0$ and checking formulas shows that $\mathbf{A}^0$ is the analogue for DGA's of the cyclic object used to define cyclic homology, compare [8, 11]. We will refer to $\mathbf{A}^0$ as the cyclic chain complex generated by A.

Example 1.5. Let X be a cyclic space and Y a cocyclic space. Then we get cyclic chain complexes $\mathbf{n} \to S_* X(\mathbf{n})$ and $\mathbf{n} \to S^* Y(\mathbf{n})$.

§ 2. Cyclic homology

We define the cyclic homology, in all its variations, of a cyclic chain complex E. This will be done by the shortest possible route, for a less abbreviated account see [8–11; 18–20]. The differential in E will be assumed to decrease degree by one. We write d_i, s_i and t_n for the maps induced by δ_i, σ_i and τ_n, so d_i, s_i and t_n satisfy the opposites of the relations 1.1.

First we construct the Hochschild complex of E. Ignoring the maps t_n, E is a simplicial chain complex; associated to such a structure is a natural double complex and the Hochschild complex $C(E)$ is the associated total complex. Explicitly form the double complex

$$C_{p,q} = C_{p,q}(E) = E(\mathbf{p})_q$$
$$b_{\mathrm{I}}: C_{p,q} \to C_{p,q-1}, \qquad b_{\mathrm{II}} = \sum (-1)^i d_i: C_{p,q} \to C_{p-1,q}$$

where b_{I} is the differential in the chain complex $E(\mathbf{p})$. The associated total complex is given by

$$C_n = C_n(E) = \bigoplus_{p+q=n} C_{p,q}(E)$$

with total differential $b(x)=b_{\mathrm{I}}(x)+(-1)^p b_{\mathrm{II}}(x)$ for x in $C_{p,q}$. The Hochschild homology of E, $HH_*(E)$, is the homology of this total complex.

Next we construct the B operator (compare [10] and [20]) B: $C_{p,q}\to C_{p+1,q}$. First define two auxilliary operators

$$h_p=t_{p+1}s_p\colon E(\mathbf{p})\to E(\mathbf{p}+\mathbf{1})$$
$$N_p=\sum_{0\leqq i\leqq p}(-1)^{ip}t_p^i\colon E(\mathbf{p})\to E(\mathbf{p}).$$

Then B is defined by the formula

$$B_p=(-1)^q(1-(-1)^{p+1}t_{p+1})h_pN_p\colon E(\mathbf{p})_q\to E(\mathbf{p}+\mathbf{1})_q.$$

A straightforward argument, compare [10, Lemma 30] and [20, 1.3 and 1.4], shows that $B^2=0$, $Bb_{\mathrm{I}}=-b_{\mathrm{I}}B$ and $Bb_{\mathrm{II}}=-b_{\mathrm{II}}B$.

If L and M are graded K modules then $L\otimes M$ will denote the tensor product defined by

$$(L\otimes M)_n=\prod_{i+j=n}L_i\otimes M_j.$$

The use of the direct product rather than the direct sum in forming tensor products of graded modules is an important technical point.

Now introduce the polynomial ring $K[u]$, where $|u|=-2$ and construct chain complexes

$$\begin{array}{ll}\mathbf{C}^-(E)=K[u]\otimes C(E) & \text{differential } \partial^-\\ \mathbf{C}^\wedge(E)=u^{-1}\mathbf{C}^-(E) & \text{differential } \partial^\wedge\\ \mathbf{C}^+(E)=\mathbf{C}^\wedge(E)/u\mathbf{C}^-(E) & \text{differential } \partial^+.\end{array}$$

The differentials are defined by the formulas

$\partial^-=b+uB$ that is $\partial^-(u^k\otimes y)=u^k\otimes by+u^{k+1}\otimes By$,

$\partial^\wedge$ is the differential induced on the localisation of $\mathbf{C}^-$

∂^+ is the differential induced by $\partial^\wedge$ on the quotient $\mathbf{C}^+$.

If we write out these chain complexes as double complexes with columns $C(E)$, horizontal differential B and vertical differential b, as in [20], then $\mathbf{C}^-$ occupies the left hand half plane, $\mathbf{C}^+$ the right hand half plane and $\mathbf{C}^\wedge$ the whole plane. Further, the total complex $\mathbf{C}^-$ is formed using the direct product and for $\mathbf{C}^+$ it is formed using the direct sum. For $\mathbf{C}^\wedge$ the general element is of the form $\sum a_nu^n$ where $n\in\mathbf{Z}$ and $a_n=0$ for $n\leqq n_0$. Note that we will have to deal with cyclic chain complexes, like the (negatively graded) cochains on a cocyclic space, with elements of arbitrarily large negative degrees. So in all cases it is important to be precise over how we form the total complex from the corresponding double complex. The definitions are chosen so that 2.1 below is valid.

The various cyclic homology groups are defined as follows:

$$HC_*^-(E)=H_*\mathbf{C}^-(E),\qquad H\hat{C}_*(E)=H_*\mathbf{C}^\wedge(E),\qquad HC_*(E)=H_*\mathbf{C}^+(E).$$

By construction $HC^-_*(E)$ and $HC_*(E)$ are modules over $K[u]$ and $H\hat{C}_*(E)$ is a module over $K[u, u^{-1}]$. By definition the periodicity operator is taken to be multiplication by u. Note that since forming homology and localisation commute $H\hat{C}_*(E)=u^{-1}HC^-_*(E)$. In particular, if A is a DGA then

$$HH_*(A)=HH_*(\mathbf{A}^0),\quad HC^-_*(A)=HC^-_*(\mathbf{A}^0),\quad H\hat{C}_*(A)=H\hat{C}_*(\mathbf{A}^0),$$
$$HC_*(A)=HC_*(\mathbf{A}^0).$$

The fundamental exact sequence is the exact sequence of homology groups induced by the short exact sequence of chain complexes

$$0\to\mathbf{C}^-(E)\to\mathbf{C}^\wedge(E)\to u\mathbf{C}^+(E)\to 0$$

after identifying $H_n(u\mathbf{C}^+(E))$ with $HC_{n-2}(E)$.

The following lemma gives one particularly important property of Hochschild and cyclic homology.

Lemma 2.1. *Let $f\colon E\to F$ be a map of cyclic chain complexes such that in each degree n, $f(\mathbf{n})\colon E(\mathbf{n})\to F(\mathbf{n})$ induces an isomorphism in homology. Then f induces isomorphisms*

(i) $HH_*(E)\to HH_*(F)$
(ii) $HC_*(E)\to HC_*(F)$
(iii) $HC^-_*(E)\to HC^-_*(F)$
(iv) $H\hat{C}_*(E)\to H\hat{C}_*(F)$.

Proof. The proof of (i), and the deduction of (ii) from (i) follow from routine double complex arguments. If $m\geq n$, write $\mathbf{C}\langle m,n\rangle$ for the subquotient $u^n\mathbf{C}^\wedge/u^m\mathbf{C}^\wedge$ of $\mathbf{C}^\wedge$. Induction starting from (i) shows that the induced map $H_*\mathbf{C}(E)\langle m,n\rangle\to H_*\mathbf{C}(F)\langle m,n\rangle$ is an isomorphism. Now by construction

$$u^m\mathbf{C}^-=\operatorname*{Inv\,lim}_{n\to\infty}\mathbf{C}\langle m,n\rangle$$

and so the $\lim^1$ exact sequence for computing the homology of an inverse limit of chain complexes shows that the induced map $H_*(u^m\mathbf{C}^-(E))\to H_*(u^m\mathbf{C}^-(E))$ is an isomorphism for all m. In particular this proves (iii). Finally (iv) follows since $H\hat{C}_*(E)=u^{-1}HC^-_*(E)$.

We end this section with an indication of the definition of the cyclic cohomology of E. Simply replace the Hochschild complex of E by its K dual $C^*(E)$, with differential b^*, and the B operator by its dual B^*. Give the indeterminate u in the polynomial ring $K[u]$ degree 2 and form the complexes $\mathbf{D}^+(E)=K[u]\otimes C^*(E)$, $\mathbf{D}^\wedge(E)=u^{-1}\mathbf{D}^+(E)$ and $\mathbf{D}^-(E)=\mathbf{D}^\wedge(E)/\mathbf{u}\mathbf{D}^+(E)$ with differentials $\partial^+=b^*+uB^*$, and $\partial^\wedge$, ∂^- the induced differentials. The various forms of cyclic cohomology are defined as follows

$$HC^*(E)=H_*\mathbf{D}^+(E),\quad H\hat{C}^*(E)=H_*\mathbf{D}^\wedge(E),\quad HC^*_-(E)=H_*\mathbf{D}^-(E).$$

Written out as double complexes (with differentials increasing degree by one) as in the case of cyclic homology, $\mathbf{D}^+$ occupies the right hand half plane

$\mathbf{D}^-$ the left hand half plane and $\mathbf{D}^\wedge$ the whole plane. Again the precise way in which we form the total complexes is important. Finally note that $H\hat{C}^*(E) = u^{-1}HC^*(E)$.

§ 3. The main theorems

Let X be a cyclic space and Y a cocyclic space. By forgetting the structure maps induced by τ_n, X becomes a simplicial space and Y a cosimplicial space. We will denote the cosimplicial space $\mathbf{n} \to \Delta^n$, where Δ^n is the standard n simplex by $\Delta^\cdot$. One may form the realisation $|X|$ of a simplicial space, see [25]. The construction of the realisation $|Y|$ of the cosimplicial space Y (see [6, § 5]) is less well known. By definition $|Y| = \mathrm{Hom}_\Delta(\Delta^\cdot, Y)$ where Hom_Δ means natural transformations of functors defined on Δ, and $|Y|$ is given the topology it inherits as a subset of $\prod \mathrm{Map}(\Delta^n, Y(\mathbf{n}))$.

We now introduce some grading and notational conventions. We always assume that the differential in a chain complex decreases degree by one. To accomodate this convention we are forced to grade cochains negatively and therefore to grade cohomology, and $H^*_{\mathbb{T}}$, negatively and change the sign of the grading of $\hat{H}^*_{\mathbb{T}}$ and $G^*_{\mathbb{T}}$. We write $S_*(X)$ for the cyclic chain complex defined by the cyclic space X. Similarily, $S^*(Y)$ will denote the cyclic chain complex determined by the cocyclic space Y. We always assume that a cyclic space, when considered as a simplicial space, is "good" in the technical sense of [25, Definition A.4]. This condition ensures that the homology of $|X|$ is naturally isomorphic to the homology of $C_*(X)$, compare [5, Theorem 4.1].

We now state the main results relating Λ and $\mathbb{T}$, cyclic homology and equivariant homology.

Theorem 3.1. *Let X be a cyclic space, then $\mathbb{T}$ acts on $|X|$ in such a way that:*

(i) *$X \to |X|$ becomes a functor from the category of cyclic spaces to the category of spaces with a circle action and equivariant maps.*

(ii) *There are natural isomorphisms*

$$HC_*(S_*X) \cong H^{\mathbb{T}}_*(|X|) \quad \text{as } K[u] \text{ modules}$$
$$H\hat{C}_*(S_*X) \cong \hat{H}^{\mathbb{T}}_*(|X|) \quad \text{as } K[u, u^{-1}] \text{ modules}$$
$$HC^-_*(S_*X) \cong G^{\mathbb{T}}_*(|X|) \quad \text{as } K[u] \text{ modules.}$$

These isomorphisms throw the fundamental exact sequence for cyclic homology onto the fundamental exact sequence for equivariant homology.

This theorem, without the HC^- statement, can deduced from [7, 13 and 14]. The precise relation between the category of cyclic sets and spaces with circle action is carefully studied in [13].

Now let Y be a cocyclic space; there is a natural chain map ψ: $C(S^*Y) \to S^*(|Y|)$ where $C(S^*Y)$ is the Hochschild complex of the cyclic chain complex $S^*(Y)$, see § 4 for a definition of ψ. We say that Y converges if ψ induces an isomorphism in homology; these matters are discussed in [6, § 7] and [2].

Theorem 3.2. *Let Y be a cocyclic space, then $\mathbb{T}$ acts on $|Y|$ in such a way that:*

(i) *$Y \to |Y|$ becomes a functor from the category of cocyclic spaces to the category of spaces with a circle action and equivariant maps.*

(ii) *If Y converges there are natural isomorphisms*

$$\begin{aligned} HC_*^-(S^*Y) &\cong H_{\mathbb{T}}^*(|Y|) && \text{as } K[u] \text{ modules} \\ H\hat{C}_*(S^*Y) &\cong \hat{H}_{\mathbb{T}}^*(|Y|) && \text{as } K[u,u^{-1}] \text{ modules} \\ HC_*(S^*Y) &\cong G_{\mathbb{T}}^*(|Y|) && \text{as } K[u] \text{ modules.} \end{aligned}$$

These isomorphisms throw the fundamental exact sequence for cyclic homology onto the fundamental exact sequence for equivariant cohomology.

Theorem 3.3. *Let Z be a space with a circle action. The function $\mathbf{n} \to S_n(Z)$ can be made into a cyclic K module $S_*(Z)$ in such a way that:*

(i) *$Z \to S_*(Z)$ becomes a functor from the category of spaces with a circle action to the category of cyclic K modules.*

(ii) *There are natural isomorphisms*

$$\begin{aligned} HC_*(S_*Z) &\cong H_*^{\mathbb{T}}(Z) && \text{as } K[u] \text{ modules} \\ H\hat{C}_*(S_*Z) &\cong \hat{H}_*^{\mathbb{T}}(Z) && \text{as } K[u,u^{-1}] \text{ modules} \\ HC_*^-(S_*Z) & G_*^{\mathbb{T}}(Z) && \text{as } K[u] \text{ modules.} \end{aligned}$$

These isomorphisms throw the fundamental exact sequence for cyclic homology onto the fundamental exact sequence for equivariant homology.

In [17] the question of products in the various cyclic homology and cohomology theories is discussed and it is proved that the isomorphisms of Theorems 3.1, 3.2 and 3.3, and therefore Theorems A and B, respect the products one can define in the cyclic theories and the equivariant theories.

To begin the proof we analyse the cyclic sets λ^n defined by $\lambda^n(\mathbf{m}) = \Lambda(\mathbf{m}, \mathbf{n})$ and their realisations $|\lambda^n| = \Lambda^n$. Note that $\mathbf{n} \to \Lambda^n$ defines a cocyclic space which will be denoted by $\Lambda^{\bullet}$. There is another cocyclic space $\mathbb{T} \times \Delta^{\bullet}$, compare [14]; the face and degeneracy maps are given by the products of the identity map of $\mathbb{T}$ with the usual face and degeneracy maps of $\Delta^{\bullet}$ and τ_n is defined by

$$\tau_n(z, u_0, \ldots, u_n) = (z \exp(-2\pi i u_0), u_1, \ldots, u_n, u_0).$$

Here $z \in \mathbb{T}$ and we have identified Δ^n with the set

$$\{(u_0, \ldots, u_n) | 0 \leqq u_i \leqq 1, \textstyle\sum u_i = 1\} \subset \mathbb{R}^{n+1}.$$

It is straightforward to check that the relations 1.1(a)–(d) hold.

Theorem 3.4. *There is an isomorphism of cocyclic spaces $\Lambda^{\bullet} \cong \mathbb{T} \times \Delta^{\bullet}$. In particular the circle acts on Λ^n in such a way that the structure maps of the cocyclic space $\Lambda^{\bullet}$ are equivariant.*

This is proved in [13]; nonetheless, for the sake of completeness and also because we will need some of the details later, we give the proof.

Proof of 3.4. Order the vertices of Δ^n as $v_0, \ldots, v_n$ where

$$v_i = (0, \ldots, 0, 1, 0, \ldots, 0) \quad 1 \text{ in the } (i+1)\text{-th place}$$

and now triangulate $\mathbb{R} \times \Delta^n$ as follows. The vertices of the triangulation are the points (i, v_r) where i is an integer. The vertices are ordered lexicographically, that is $(i, v_r) < (j, v_s)$ if either $i < j$ or $i = j$ but $r < s$. The q simplices of the triangulation are of two types, first those with vertices

$$(i, v_{r_s}), (i, v_{r_{s+1}}), \ldots, (i, v_{r_q}), (i+1, v_{r_0}), (i+1, v_{r_1}), \ldots, (i-1, v_{r_{s-1}})$$

where $r_0 < r_1 < \ldots < r_q$ and then those with vertices

$$(i, v_{r_s}), (i, v_{r_{s+1}}), \ldots, (i, v_{r_{q-1}}), (i+1, v_{r_0}), (i+1, v_{r_1}), \ldots, (i+1, v_{r_s})$$

where $r_0 < r_1 < \ldots < r_{q-1}$. Diagram 1 shows the triangulation of $[0, 2] \times \Delta^2 \subset \mathbb{R} \times \Delta^2$.

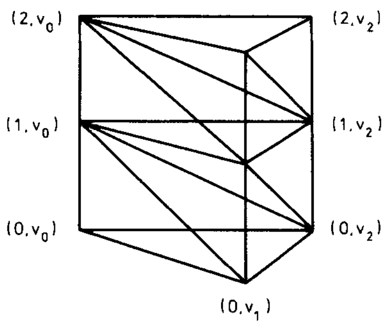

Diagram 1

Let Σ^n be the simplicial set generated by this triangulation of $\mathbb{R} \times \Delta^n$ and define an operation β_q on the q simplices of Σ^n as follows. Suppose that the last vertex of σ is (i, v_k), then the vertices of $\beta_q \sigma$ are the same as the vertices of σ except that (i, v_k) is replaced by $(i-1, v_k)$. It is clear that $(i-1, v_k)$ now becomes the first vertex of $\beta_q \sigma$. One can easily check that $d_i \beta_q = \beta_{q-1} d_{i-1}$, $d_0 \beta_q = d_q$, $s_i \beta_q = \beta_{q+1} s_{i-1}$ and finally $s_0 \beta_q = \beta_{q+1}^2 s_q$ where the operations d_i, s_i are the usual face and degeneracy operations. Therefore the operations d_i, s_i and β_n satisfy the opposites of the relations 1.1 (a)–(e) but not 1.1 (d) that is $\beta_q^{q+1} \neq 1$, in fact β_q^{q+1} simply takes a q simplex of $\mathbb{R} \times \Delta^n$ and translates it by -1.

Now note that the $n+1$ simplex with vertices

$$(i, v_r), (i, v_{r+1}), \ldots, (i, v_n), (i+1, v_0), \ldots, (i+1, v_{r-1}), (i+1, v_r)$$

is the simplex $\beta_{n+1}^{-i(n+2)} s_n \beta_n^{n-r} \iota_n$; since every simplex in the triangulation of $\mathbb{R} \times \Delta^n$ is a face of such a simplex the simplicial set Σ^n is generated, using the operations d_i, s_i and β_q, by the simplex $\iota_n = 0 \times \Delta^n$. The only relations between these operations are the relations 1.1 (a)–(e). The simplicial set λ^n is generated using the operations d_i, s_i and t_q by the identity map in $\Lambda(\mathbf{n}, \mathbf{n})$ but these operations satisfy all the relations 1.1. Therefore λ^n is obtained from Σ^n by identifying β_q^{q+1} with the identity. It now follows that Λ^n is obtained from $|\Sigma^n| = \mathbb{R} \times \Delta^n$ by identifying two points if their $\mathbb{R}$ coordinates differ by an integer. This proves that Λ^n is homeomorphic to $\mathbb{T} \times \Delta^n$.

It remains to check that this does in fact give a homeomorphism of $\mathbb{T} \times \Delta^{\cdot}$ with $\Lambda^{\cdot}$ as cocyclic spaces. But note that a map of cyclic sets $\lambda^n \to \lambda^m$ is completely determined by its value on the identity map $\iota_n \in \Lambda(\mathbf{n}, \mathbf{n})$ and it is not too difficult to check that the structure maps of $\mathbb{T} \times \Delta^{\cdot}$ as a cocyclic space map simplices to simplices and are linear on each simplex. So by determining their effect on $0 \times \Delta^n$ one can check that they correspond, under the above homeomorphism, to maps induced by appropriate maps of cyclic sets.

Lemma 3.5. *The simplices* $t_{n+1} s_n t_n^i \iota_n$, $0 \leqq i \leqq n$, *are the nondegenerate* $n+1$ *simplices of* λ^n.

Proof. This is a simple consequence of the proof of 3.4.

We can now describe, in the notation of 3.1, 3.2 and 3.3, the action of $\mathbb{T}$ on $|X|$, $|Y|$, the cyclic structure of $S_*(Z)$ and so prove the first statements in 3.1, 3.2 and 3.3. If X is a cyclic space, then by definition $|X|$ is the space

$$\coprod X(\mathbf{n}) \times \Delta^n / (\varphi^* x, t) \equiv (x, \varphi_* t)$$

where φ runs through the morphisms in Δ. However using the cyclic structure of X we see that $|X|$ is the same as

$$\coprod X(\mathbf{n}) \times \Lambda^n / (\theta^* x, t) \equiv (x, \theta_* t)$$

where now θ runs through the morphisms in Λ. However, from 3.4, $\mathbb{T}$ acts on Λ^n in such a way that the cocyclic structure maps θ_* are equivariant and so we get an induced action on $|X|$.

To give the action of $\mathbb{T}$ on the realisation of a cocyclic space Y we argue firstly that $\mathrm{Hom}_\Delta(\Delta^{\cdot}, Y) = \mathrm{Hom}_\Lambda(\Lambda^{\cdot}, Y)$ and secondly that since $\mathbb{T}$ acts on Λ^n in such a way that the cocyclic structure maps are equivariant $\mathrm{Hom}\ (\Lambda^{\cdot}, Y)$ is a $\mathbb{T}$ invariant subspace of $\prod \mathrm{Map}(\Lambda^n, X(\mathbf{n}))$ where the action of $\mathbb{T}$ on the mapping space is given by its action on Λ^n.

Finally to give the cyclic structure of $S_*(Z)$ note that using the $\mathbb{T}$ action on Z we see that $\mathrm{Map}(\Delta^n, Z) = \mathrm{Map}_{\mathbb{T}}(\mathbb{T} \times \Delta^n, Z)$ and since the cocyclic structure maps of $\mathbb{T} \times \Delta^n$ are $\mathbb{T}$ equivariant we get a cyclic structure on $\mathrm{Map}(\Delta^n, Z)$ and therefore on $S_*(Z)$.

§ 4. The *B* operator

As in § 3 let X be a cyclic space, Y a cocyclic space and Z a space with an action of $\mathbb{T}$. We get cyclic chain complexes $S_*(X)$ and $S^*(Y)$ and, from 3.1, a cyclic K module $S_*(Z)$. In this section we give a geometric interpretation of the B operators in these three cyclic objects in terms of the $\mathbb{T}$ action on the spaces $|X|$, $|Y|$ and Z. Let U and V be spaces; from now on, we write

$\theta\colon S_*(U)\otimes S_*(V)\to S_*(U\times V)$ for the chain equivalence defined by the Eilenberg McLane shuffle product.

Let W be a space with a circle action $f\colon \mathbb{T}\times W\to W$. Then $\theta\colon S_*(\mathbb{T})\otimes S_*(W)\to S_*(\mathbb{T}\times W)$ and the map f give chain level operations

$$I\colon S^{-n}(W)\to S^{-n+1}(W),\qquad J\colon S_n(W)\to S_{n+1}(W)$$

defined by the formulas $I(x)=(-1)^{|x|}f^*x/z$, $J(x)=(-1)^{|x|}f_*\theta(z\otimes x)$ where z is the fundamental 1-cycle in $S_1(\mathbb{T})$. The slant product, or integration, operation is defined by the formula $\langle a/z, x\rangle=\langle a, \theta(z\otimes x)\rangle$ for $a\in S^*(\mathbb{T}\times W)$. Since z is a cycle $\delta I=-I\delta$ and $dJ=-Jd$. From the explicit formula for θ one can check that $I^2=J^2=0$.

Let $C_*(X)$ and $C_*(Y)$ be the Hochschild complexes of the cyclic chain complexes $S_*(X)$ and $S^*(Y)$ respectively and continue to write $S_*(Z)$ for the Hochschild complex of the cyclic K module $S_*(Z)$. There are natural chain maps $\varphi\colon C_*(X)\to S_*(|X|)$ and $\psi\colon C_*(Y)\to S^*(|Y|)$ and the main purpose of this section is to prove the following result:

Theorem 4.1. *There are natural maps $h\colon C_*(X)\to S_*(|X|)$, $j\colon C_*(Y)\to S^*(|Y|)$ and $k\colon S_*(Z)\to S_*(Z)$ which raise degree by two and satisfy the formulas $dh-hb=J\varphi-\varphi B$, $\delta j-jb=I\psi-\psi B$ and $dk-kd=J-B$.*

The essential meaning of this theorem is that the following diagrams "commute up to a natural chain homotopy":

$$\begin{array}{ccc} C_*(X) & \xrightarrow{\ \varphi\ } & S_*(|X|) \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle J} \\ C_{*+1}(X) & \xrightarrow[\ \varphi\ ]{} & S_{*+1}(|X|) \end{array}\qquad \begin{array}{ccc} C_*(Y) & \xrightarrow{\ \psi\ } & S^*(|Y|) \\ {\scriptstyle B}\downarrow & & \downarrow{\scriptstyle I} \\ C_{*+1}(Y) & \xrightarrow[\ \psi\ ]{} & S^{*+1}(|Y|), \end{array}$$

and that there is a natural "chain homotopy" between the maps B and J defined on $S_*(Z)$. However this is not quite accurate since neither B, J nor I are chain maps, $Bb=-bB$, $Jd=-dJ$ and $I\delta=-\delta I$.

We begin the proof by describing the chain map φ. By the construction of the realisation of a simplicial space there are maps $\pi_n\colon X(\mathbf{n})\times\Delta^n\to|X|$. If $x\in S_q(X(\mathbf{n}))$ then $\varphi(x)=\pi_{n*}\theta(x\otimes\kappa_n)$ where $\kappa_n\in S_n(\Delta^n)$ is the fundamental n simplex. Let δ_i be the inclusion of the i-th face of Δ^n so $d\kappa_n=\sum(-1)^i\delta_{i*}\kappa_{n-1}$. From the identifications in the construction of $|X|$, $\pi_{n*}\theta(x\otimes\delta_{i*}\kappa_{n-1})=(\pi_{n-1})_*\theta(d_{i*}x\otimes\kappa_{n-1})$ where $d_i\colon X(\mathbf{n})\to X(\mathbf{n-1})$ is the structure map of X. It follows that $\varphi b_{\mathrm{II}}(x)=\pi_{n*}\theta(x\otimes d\kappa_n)$ and it is now easy to check that φ is a chain map $C_*(X)\to S_*(|X|)$. Using the cyclic structure of X these maps π_n extend to maps $\rho_n\colon X(\mathbf{n})\times\Lambda^n\to|X|$ and, from the definition of the $\mathbb{T}$ action on $|X|$, the ρ_n are equivariant. Therefore if $x\in S_q(X(\mathbf{n}))$, $\varphi(x)=\rho_{n*}\theta(x\otimes\iota_n)$ where $\iota_n\in S_n(\Lambda^n)$ is the fundamental n simplex and $\varphi B(x)=\rho_{n*}\theta(x\otimes B\iota_n)$.

The most important observation is that from 3.5 and the definition of the B operator we see that in the simplicial set λ^n, or geometrically in Λ^n, $B\iota_n$ is the fundamental class modulo degenerate $n+1$ simplices. This is the element which is the sum of the nondegenerate $n+1$ simplices, each with a sign attached and the signs are chosen so that when we take the boundary, common faces cancel in pairs. Now regard Λ^n as a space with $\mathbb{T}$ action, then $J(\iota_n)$ will give another fundamental class in $S_{n+1}(\Lambda^n)$. It follows that in general, modulo degenerate simplices, $\varphi B(x)$ and $J(\varphi x)$ represent the same geometric chain but this chain is divided up into simplices in different ways. A routine acyclic models argument (see [16, 29.95] for the method of acyclic models), which we outline below, proves Theorem 4.1; essentially up to chain homotopy this makes no difference.

We start by proving 4.1 in the case where X is a cyclic set, that is a discrete cyclic space. Suppose that we have defined $h(x)$ for all elements x of total degree $<n$ then we will define h on elements of degree n by induction. By naturality it is sufficient to define h on the universal example $\iota_n \in C_n(\lambda^n)$. From our inductive hypothesis we have defined $Q = hb(\iota_n) + J\varphi(\iota_n) - \varphi B(\iota_n)$ and Q is a cycle of degree $n+1$ in $S_*(\Lambda^n)$. Provided $n>0$ the homology of $S_*(\Lambda^n)$ is zero in degree $n+1$ so this cycle is a boundary and we can define $h(\iota_n)$ by choosing an element whose boundary is Q. If $n=0$ then it is easy to check directly that Q is a boundary and so we may start the induction.

The proof when X is a cyclic space is identical except that we must use the elements $\theta(\kappa_k \otimes \iota_n) \in S_*(\Delta^k \times \Lambda^n)$ as universal examples.

Now let Y be a cocyclic space; we first describe the map $\psi\colon C_*(Y) \to S^*(|Y|)$. The inclusion $|Y| \to \prod \mathrm{Map}(\Delta^n, Y(\mathbf{n}))$ gives maps $\alpha_n\colon \Delta^n \times |Y| \to Y(\mathbf{n})$ and, for $x \in S^q(Y(\mathbf{n}))$, $\psi(x) = \alpha_n^*(x)/\kappa_n$, where κ_n is the fundamental n simplex in $S_n(\Delta^n)$. The following diagram commutes

$$\begin{array}{ccc} \Delta^n \times |Y| & \xrightarrow{\alpha_n} & Y(\mathbf{n}) \\ \uparrow{\scriptstyle s\times 1} & & \uparrow{\scriptstyle s} \\ \Delta^m \times |Y| & \xrightarrow[\alpha_m]{} & Y(\mathbf{m}) \end{array}$$

where s is any morphism in Δ. Therefore $\psi(\delta_i^* x) = \alpha_{n-1}^* \delta_i^* x/\kappa_{n-1} = \alpha_n^* x/\delta_{i*}\kappa_{n-1}$. It now follows that $\psi(b_1 x) = \alpha_n^* x/d\kappa_n$. We use the following sign convention for the coboundary operator δ on singular cochains: $\langle \delta c, x\rangle = (-1)^{|c|+1}\langle c, dx\rangle$, compare [22, p. 258]. This leads to the formula $\delta(x/a) = \delta x/a + (-1)^{|x|} x/da$. It now follows that $\psi(bx) = \delta\psi(x)$, that is ψ is a chain map.

Using the cocyclic structure of Y and the definition of the $\mathbb{T}$ action on $|Y|$ these maps α_n extend to maps $\beta_n\colon \Lambda^n \times |Y| \to Y(\mathbf{n})$ and so $\psi(x) = \beta_n^*(x)/\iota_n$ where ι_n is the fundamental n simplex in $S_n(\Lambda^n)$. Furthermore the following diagram commutes

$$\begin{array}{ccc} \Lambda^n \times |Y| & \xrightarrow{\beta_n} & Y(\mathbf{n}) \\ \uparrow{\scriptstyle t\times 1} & & \uparrow{\scriptstyle t} \\ \Lambda^m \times |Y| & \xrightarrow[\beta_m]{} & Y(\mathbf{m}) \end{array}$$

where t is any morphism in Λ. The above argument gives, for any cyclic set, a natural chain homotopy h such that $dh - hb = \varphi B - J\varphi$ and, using this homotopy in λ^n, we define a map $j\colon C_*(Y) \to S^*(|Y|)$ by the formula $j(x)$

$=\beta_n^* x/h(\iota_n)$ if $x \in S^q(Y(\mathbf{n}))$. We now compute $\delta j(x) - jb(x)$:

$$\delta j(x) = \beta_n^*(\delta x)/h(\iota_n) + (-1)^q \beta_n^*(x)/dh(\iota_n)$$
$$jb(x) = \beta_{n-1}^*(b_{\mathrm{II}} x)/h(\iota_{n-1}) + (-1)^q \beta_n^*(\delta x)/h(\iota_n).$$

From the above diagram, $\beta_{n-1}^*(b_{\mathrm{II}}x)/h(\iota_{n-1}) = \sum (-1)^i \beta_n^*(x)/\delta_{i*}h(\iota_{n-1})$. However h is natural with respect to maps of cyclic sets so $\delta_{i*}h(\iota_{n-1}) = h(\delta_{i*}\iota_{n-1})$ and therefore $\beta_{n-1}^*(b_{\mathrm{II}}x)/h(\iota_{n-1}) = \beta_n^*(x)/h(d\iota_n)$. Therefore we have proved that $\delta j(x) - jb(x) = (-1)^q \beta_n^* x/J\varphi(\iota_n) - (-1)^q \beta_n^* x/\varphi B(\iota_n)$. Using 4.2 and the definition of B (taking care over the signs in the definition of B) one can check that $\beta_n^* x/\varphi B(\iota_n) = (-1)^q \psi B(x)$. From the definition of the action of $\mathbb{T}$ on $|Y|$ one can check that the map $\beta_n\colon \Lambda^n \times |Y| \cong \mathbb{T} \times \Delta^n \times |Y| \to Y(\mathbf{n})$ is the composite of first the map defined by the $\mathbb{T}$ action and then the map α_n. It now follows (taking care of the sign introduced by switching $\mathbb{T}$ and Δ^n so that we may first apply the map defined by the $\mathbb{T}$ action) that $\beta_n^* x/J\varphi(\iota_n) = (-1)^q I\psi(x)$ and so $\delta j(x) - jb(x) = I\psi(x) - \psi B(x)$. This gives the natural chain homotopy.

The proof of (ii) is a straightforward modification of the proof of (i) in the case of cyclic sets.

We close this section with a final observation concerning the B operator. If we make the circle act on $\mathbb{T} \times \Delta^n$ by $z(w,t) = (z^{-1}w, t)$ and give $|X|$ the induced $\mathbb{T}$ action and corresponding J operator, then one can check that, modulo degenerate simplices, $\varphi B = -J\varphi$. We have chosen not to exploit this fact in the proof of 4.1 simply to make it clear that the result does not in any way depend on the precise choice of the chain equivalence θ.

§ 5. Equivariant homology and cohomology theories

Let W be a space with an action of the circle $f\colon \mathbb{T} \times W \to W$. Introduce the polynomial ring $K[u]$ where u has degree -2 and form chain complexes

$$\begin{array}{ll} \mathbf{U}^-(W) = K[u] \otimes S_*(W) & \text{differential } \partial^- \\ \mathbf{U}^\wedge(W) = u^{-1}\mathbf{U}^-(W) & \text{differential } \partial^\wedge \\ \mathbf{U}^+(W) = \mathbf{U}^\wedge(W)/\mathbf{u}\mathbf{U}^-(W) & \text{differential } \partial^+. \end{array}$$

The differential ∂^- is defined by the formula $\partial^- = d + uJ$ and ∂^+, $\partial^\wedge$ are the induced differentials.

Lemma 5.1. *There is a natural isomorphism of $H_*(\mathrm{U}^+ W)$ with $H_*^{\mathbb{T}}(W)$.*

Proof. It is not too difficult to convince oneself that the chain complex $\mathbf{U}^+(W)$ is in fact an explicit model for the chains on $E\mathbb{T} \times_{\mathbb{T}} W$. One proof is to use the fact that $E\mathbb{T} \times_{\mathbb{T}} W$ is the geometrical realisation of the following simplicial space; in degree n we put $\mathbb{T}^n \times W$, the face maps are given by

$$d_0(t_1, \dots, t_n, w) = (t_2, \dots, t_n, w)$$
$$d_i(t_1, \dots, t_n, w) = (t_1, \dots, t_i t_{i+1}, \dots, t_n, w) \qquad 1 \leqq i \leqq n-1$$
$$d_n(t_1, \dots, t_n, w) = (t_1, \dots, t_{n-1}, t_n w)$$

and the degeneracy maps are given by inserting the unit of $\mathbb{T}$. Let M_* be the total complex of the simplicial chain complex $\mathbf{n} \to S_*(\mathbb{T}^n \times W)$ so that the homology of M_* is $H_*(E\mathbb{T} \times_{\mathbb{T}} W)$. The chain equivalence θ and the multiplication map of $\mathbb{T}$ make $S_*(\mathbb{T})$ into a strictly graded commutative (that is the square of an element of odd degree is zero) DGA and $S_*(W)$ becomes a module over $S_*(\mathbb{T})$. There is a map from (the total complex of) the bar complex for $S_*(W)$ over $S_*(\mathbb{T})$ to M_* which is an isomorphism in homology. The inclusion of the exterior subalgebra $E(z) \to S_*(\mathbb{T})$, where z is the fundamental 1 simplex, is a chain equivalence. Now make $S_*(W)$ into a module over $E(z)$ by setting $zx = Jx$ so there is a map of the bar complex of $S_*(W)$ over $E(z)$ into M_* which is an isomorphism in homology. Finally, there is a canonical equivalence of the bar complex of $S_*(W)$ over $E(z)$ with the complex $\mathbf{U}^+(W)$.

We now define the equivariant homology theories:

$$H_*^{\mathbb{T}}(W) = H_*(\mathbf{U}^+(W)), \qquad \hat{H}_*^{\mathbb{T}}(W) = H_*(\mathbf{U}^\wedge(W)), \qquad G_*^{\mathbb{T}}(W) = H_*(\mathbf{U}^-(W)).$$

In view of Lemma 5.1, this definition of $H_*^{\mathbb{T}}(W)$ agrees with that given in the introduction. Since forming homology and localisation commute, $\hat{H}_*^{\mathbb{T}}(W) = u^{-1} G_*^{\mathbb{T}}(W)$.

To define the equivariant cohomology theories we use chain complexes $\mathbf{V}^-(W)$, $\mathbf{V}^\wedge(W)$ and $\mathbf{V}^+(W)$ obtained by replacing $S_*(W)$ by $S^*(W)$ and J by I. The definitions are as follows (recall the grading conventions introduced in § 3):

$$H^*_{\mathbb{T}}(W) = H_*(\mathbf{V}^-(W)), \qquad \hat{H}^*_{\mathbb{T}}(W) = H_*(\mathbf{V}^\wedge(W)), \qquad G^*_{\mathbb{T}}(W) = H_*(\mathbf{V}^+(W)).$$

As before we easily deduce that $\hat{H}^*_{\mathbb{T}}(W) = u^{-1} H^*_{\mathbb{T}}(W)$. Note that if we write out $\mathbf{V}^-(W)$ as a double complex with columns copies of $S^*(W)$, vertical differential δ and horizontal differential I, then $\mathbf{V}^-(W)$ lies in the purely negative quadrant and so $H^*_{\mathbb{T}}(W)$ is concentrated in negative degrees. The argument given in 5.1 shows that $H^*_{\mathbb{T}}(W)$ is isomorphic to $H^*(E\mathbb{T} \times_{\mathbb{T}} W)$, negatively graded.

We now describe some of the formal properties of these equivariant theories. The module structure over the coefficient rings is clear from the definition. One can check that in the cases of $H_*^{\mathbb{T}}(W)$ and $H^*_{\mathbb{T}}(W)$ this module structure coincides with the action of $H^*(B\mathbb{T})$ on $H_*(E\mathbb{T} \times_{\mathbb{T}} W)$ and $H^*(E\mathbb{T} \times_{\mathbb{T}} W)$ defined using the slant product and the cup product respectively. The fundamental exact sequences come from the short exact sequences of chain complexes

$$0 \to \mathbf{U}^-(W) \to \mathbf{U}^\wedge(W) \to u\mathbf{U}^+(W) \to 0, \qquad 0 \to \mathbf{V}^-(W) \to \mathbf{V}^\wedge(W) \to u\mathbf{V}^+(W) \to 0.$$

These equivariant theories have the following strong invariance property.

Lemma 5.2. *Let f: $W_1 \to W_2$ be an equivariant map which induces an isomorphism in ordinary homology and cohomology. Then f induces an isomorphism in each of the above theories.*

Proof. The proof is completely analogous to the proof of 2.1.

We now begin the proofs of 3.1, 3.2 and 3.3. Let X be a cyclic space then we write $\mathbf{C}^-(X)$, $\mathbf{C}^\wedge(X)$ and $\mathbf{C}^+(X)$ for the chain complexes of § 3 defined by the cyclic chain complex $S_*(X)$. The idea in the proof of 3.1 is that we know we can replace $S_*(|X|)$ by the Hochschild complex $C_*(X)$ (we are assuming that a simplicial space is automatically good so that the chain map φ of § 4 induces an isomorphism in homology) and by 4.1 we should be able to replace J by B without altering homology. However the problem we must overcome is that we do not yet have a natural chain map between the $\mathbf{C}$ complexes and the $\mathbf{U}$ complexes.

Lemma 5.3. (i) *There is a natural* $K[u]$ *module chain map* $\zeta\colon \mathbf{C}^-(X)\to \mathbf{U}^-(|X|)$ *such that the induced map* $\mathbf{C}^-(X)/u\mathbf{C}^-(X)=C_*(X)\to \mathbf{U}^-(|X|)/u\mathbf{U}^-(|X|)=S_*(|X|)$ *is the chain map* φ.

(ii) *Any two natural chain maps* ζ_1 *and* ζ_2 *satisfying the conditions in* (i) *are naturally chain homotopic.*

Proof. Pick a natural $K[u]$ module map h such that $dh-hb=J\varphi-\varphi B$, see 4.1. As a $K[u]$ module, $\mathbf{C}^-(X)$ is free with basis $C_*(X)$ and therefore it is sufficient to construct ζ on $C_*(X)$. For $x\in C_*(X)$ write $\zeta(x)=\sum \zeta_n(x)u^n$ where $\zeta_n\colon C_*(X)\to S_*(|X|)$ is a K linear map which raises degree by $2n$. In order that the $K[u]$ linear extension of ζ be a chain map the ζ_n must satisfy the following formula:

$$\zeta_n b+\zeta_{n-1}B=d\zeta_n+J\zeta_{n-1}. \tag{5.4}$$

We construct ζ_n inductively starting from $\zeta_0=\varphi$ and $\zeta_1=h$. Suppose that $n\geqq 2$ and we have defined ζ_i for $i<n$. We then suppose that we have constructed ζ_n on elements of degree $<m$ and inductively construct ζ_n on elements of degree m.

First we deal with the case where X is a cyclic set so, by naturality, it is sufficient to construct ζ_n on the element $\iota_m\in C_m(\lambda^m)$. Now by our inductive hypotheses we have determined the element $w=\zeta_n b\iota_m+\zeta_{n-1}B\iota_m-J\zeta_{n-1}\iota_m$ and $dw=0$. However the degree of w is $m-1+2n$ and therefore, since $n\geqq 2$ and the homology of $S_*(\Lambda^n)$ is zero in degrees $\geqq 2$, w must be a boundary. Therefore we can define $\zeta_n(\iota_m)$ by choosing an element whose boundary is w. This process will serve to define ζ_n on elements of degree zero and so begin the induction. The proof when X is a cyclic space is identical except that we must replace the cyclic sets λ^m by the cyclic spaces $\lambda^m\times\Delta^k$. The proof that any two choices of chain equivalence ζ are chain homotopic is almost identical.

Proof of 3.1. Let ζ be the chain map constructed in 5.5; ζ extends to a chain map $\mathbf{C}^\wedge(X)\to\mathbf{U}^\wedge(|X|)$ and induces a chain map $\mathbf{C}^+(X)\to\mathbf{U}^+(|X|)$. These chain maps induce isomorphisms in homology by the argument of 2.1.

Now let Y be a cocyclic space, then we write $\mathbf{C}^-(Y)$, $\mathbf{C}^\wedge(Y)$ and $\mathbf{C}^+(Y)$ for the chain complexes of § 3 defined by the cyclic chain complex $S^*(Y)$.

Lemma 5.5. (i) *There is a natural* $K[u]$ *module chain map* $\xi\colon\mathbf{C}^-(Y)\to\mathbf{V}^-(|Y|)$ *such that the induced map* $\mathbf{C}^-(Y)/u\mathbf{C}^-(Y)=C_*(Y)\to\mathbf{V}^-(|Y|)/u\mathbf{V}^-(|Y|)=S^*(|Y|)$ *is the chain map* ψ.

(ii) *Any two natural chain maps* ζ_1 *and* ζ_2 *satisfying the conditions in* (i) *are naturally chain homotopic.*

Proof. As in the proof of 5.3 we need maps $\xi_n\colon C_*(Y)\to S^*(|Y|)$ which raise degree by $2n$ and satisfy the equation

$$\xi_n b+\xi_{n-1}B=d\xi_n+I\xi_{n-1}. \tag{5.6}$$

The construction of the ξ_n and the proof that they satisfy 5.6 proceeds by analogy with 4.1 in the case of a cocyclic space. Pick natural maps ζ_n as in 5.3 and define ξ_n on $S^*(Y(\mathbf{m}))\subset C_*(Y)$ by the formula $\xi_n(x)=\beta_m^* x/\zeta_n(\iota_m)$. A straightforward argument completes the proofs.

Proofs of 3.2 and 3.3. The deduction of 3.2 from 5.5 is identical to the deduction of 3.1 from 5.3 except that we must use the convergence hypothesis on the cocyclic space Y to deduce that the chain maps given by ξ induce isomorphisms in homology. The proof of 3.3 is an easy modification of the proof of 3.1.

§ 6. The proofs of theorems A and B

The main ingredients in the proofs are the following results.

Theorem 6.1. *Let* X *be a topological space and let* $\mathbf{X}^0$ *be the cocyclic space of example 1.2. Then there is an equivariant homeomorphism* $|\mathbf{X}^0|\to LX$.

Theorem 6.2. *Let* G *be a topological group and let* $\mathbf{G}$ *be the cyclic space of example 1.3. Then there is an equivariant map* $|\mathbf{G}|\to LBG$ *which is an ordinary homotopy equivalence.*

Theorem 6.2 is also proved in [14, § V], and also in [7]. We also need the following technical lemma.

Lemma 6.3. *If* X *is simply connected then* $\mathbf{X}^0$ *converges.*

Proofs of Theorems A and B given 6.1, 6.2 and 6.3. By 6.1, 3.2 and 6.3 we know that $HC_*^-(S^*\mathbf{X}^0)\cong H_{\mathbb{T}}^*(LX)$ and so on. However using the Alexander Whitney chain equivalence $S^*(U)\otimes S^*(V)\to S^*(U\times V)$ we get a map of cyclic chain complexes from the cyclic chain complex generated by $S^*(X)$ (see Example 1.4) to $S^*(\mathbf{X}^0)$ which, from 2.1, is an isomorphism in all forms of cyclic homology. This proves Theorem A.

To prove Theorem B use 6.2, 3.1 and an almost identical argument.

We now prove 6.1. Let E be a cyclic set and X a topological space, and define a cocyclic space X^E by setting $X^E(\mathbf{n})=\mathrm{Map}(E(\mathbf{n}),X)$ with the obvious cocyclic structure. By 3.1 and 3.2 there are $\mathbb{T}$ actions on the spaces $|E|$ and $|X^E|$.

Lemma 6.4. *There is a natural equivariant homeomorphism between the spaces* $|X^E|$ and $\mathrm{Map}(|E|,X)$.

Proof. This is almost a tautology and is left to the reader.

Proof of 6.1. Simply observe that $\mathbf{X}^0$ is X^E where $E=\lambda^0$, and use 3.4.

Proof of 6.3. See [2] and [6, § 5].

We now start on the proof of 6.2. Let S be a simplicial set satisfying the Kan condition [21]. Let $Y_n = \mathrm{Hom}_\Delta(\lambda^n, S)$ and make $\mathbf{n} \to Y_n$ into a cyclic set Y using the maps $\lambda^n \to \lambda^m$ induced by morphisms $\mathbf{n} \to \mathbf{m}$ in Λ.

Lemma 6.4. *There is an equivariant map $|Y| \to L(|S|)$ which is an ordinary homotopy equivalence.*

Proof. We construct maps $Y_n \times \Delta^n \to L(|S|)$ as follows: Given $f \in Y_n$ then f defines a map $|f|: \Lambda^n \to |S|$. From 3.4, Λ^n can be identified with $\mathbb{T} \times \Delta^n$ and the loop in $|S|$ corresponding to (f, t) is given by $z \to |f|(z, t)$. It can be checked that these maps induce an equivariant map from $|Y| \to L(|S|)$. It is a standard fact from the theory of function complexes for simplicial sets [21], that this map is an ordinary homotopy equivalence.

Proof of 6.2 when G is a discrete group. We define BG to be the realisation of the simplicial set $B.G$ defined as follows: In degree n we put the set G^n; the face maps are given by the following formulas:

$$d_0(g_1, \ldots, g_n) = (g_2, \ldots, g_n)$$
$$d_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) \qquad 1 \leqq i \leqq n-1$$
$$d_n(g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1}).$$

The degeneracy maps are given by inserting the unit. From the proof of 3.4 we see that λ^n is generated, using the face and degeneracy operations, by the $n+1$ dimensional simplices $t_{n+1} s_n t_{n+1}^i \iota_n$, $0 \leqq i \leqq n$, and therefore a map $\lambda^n \to B.G$ of simplicial sets is determined by its values on these simplices. Taking account of the explicit form of the structure maps of λ^n and $B.G$ we find that a simplicial map between them is determined by its value on $t_{n+1} s_n \iota_n$ and this can be any element of G^{n+1}. One now checks that the cyclic spaces $\mathbf{G}$ and $\mathrm{Map}_\Delta(\lambda^n, B.G)$ are isomorphic. Since $B.G$ is satisfies the Kan condition, Lemma 6.4 completes the proof.

Proof of 6.2 in the general case. We will need to use the following well known fact about bisimplicial sets $S..$: The realisations of the simplicial spaces $n \to |S_n.|$ and $m \to |S._m|$ are naturally homeomorphic and are in turn homeomorphic to the realisation of the simplicial set $n \to S_{n,n}$. We will use $\|S..\|$ to denote this space. For example if $H.$ is a simplicial group then we can naturally associate to $H.$ the following versions of its classifying space, a bisimplicial set $B.H.$, a simplicial space $BH.$ and since $|H.|$ is a topological group a space, $B|H.|$. There are homeomorphisms of $\|B.H.\|$ with both $B|H.|$ and the realisation of the simplicial space $BH.$.

To deal with the general case of 6.2 we first observe that if G is a topological group then there is a simplicial group $H.$ and a homorphism of groups $G \to |H.|$ which is a homotopy equivalence; therefore BG and $B|H.|$ are homotopy equivalent. Now define $L_{n,m}$ to be $\mathrm{Hom}_\Delta(\lambda^n, B.H_m)$ so that $L..$ becomes a bisimplicial set. From 6.4 there is a homotopy equivalence

$|L._m| \to L(BH_m)$. Next we invoke a theorem of Anderson [3] to tell us that since BH_m is a connected space for each m, the natural map $|L(BH.)| \to L|BH.| = LB|H.|$ is an equivalence. Each of the spaces $L(BH_m)$ have $\mathbb{T}$ actions and the structure maps of the simplicial space $L(BH.)$ are $\mathbb{T}$ equivariant so $|L(BH.)|$ has a natural action of $\mathbb{T}$ and the natural map $|L(BH.)| \to LB|H.|$ is equivariant. Now since BG and $B|H.|$ are homotopy equivalent there is an equivariant homotopy equivalence $LB|H.| \to LBG$. To sum up we have constructed a $\mathbb{T}$ equivariant map $\|L..\| \to LBG$ which is a homotopy equivalence.

Now use the special case of 6.2 proved above to identify $L_{n,m}$ with $(H_m)^{n+1}$ and since realisations commute with products $|L_n.| = |H.|^{n+1}$. Therefore $n \to |L_n.|$ is the cyclic space generated by the topological group $|H.|$. The map $G \to |H.|$ gives a map from $\mathbf{G}$, the cyclic space generated by G, to the cyclic space generated by $|H.|$ which is a homotopy equivalence at each level and therefore gives a homotopy equivalence of realisations. Therefore we get a map $|\mathbf{G}| \to \|L..\|$ which is a homotopy equivalence. Since this map is defined by a map of cyclic spaces it is automatically $\mathbb{T}$ equivariant.

In total we have constructed a $\mathbb{T}$ equivariant map $|\mathbf{G}| \to LBG$ which is equivariant and a homotopy equivalence. This proves 6.2.

§ 7. An application of Theorem A

Let X be a simply connected manifold and suppose that we have taken coefficients in $\mathbb{R}$ or $\mathbb{C}$. Then in Theorem A we may replace the singular cochain algebra of X by the de Rham algebra of differential forms $\Omega^*(X)$ and then use the methods of rational homotopy theory [23, 26, 12, 15]. The notion of equivalence between commutative DGA's is the equivalence relation generated by the relation $A \sim B$ if there is a map of DGA's $A \to B$ which is a chain homotopy equivalence. A commutative DGA is said to be formal if it is equivalent to its homology algebra. A simply connected manifold is said to be formal if its de Rham algebra is equivalent to its cohomology algebra. Examples of formal manifolds are simply connected compact symmetric spaces [4, 15] and simply connected compact Kahler manifolds [12].

Theorem 7.1. *Suppose X is a simply connected formal manifold. Then, using the grading conventions of § 3, there are isomorphisms*

$$HC_*^-(H^*X) \cong H_{\mathbb{T}}^*(LX) \quad \text{as } K[u] \text{ modules}$$
$$H\hat{C}_*(H^*X) \cong H_{\mathbb{T}}^*(LX) \quad \text{as } K[u, u^{-1}] \text{ modules}$$
$$HC_*(H^*X) \cong G_{\mathbb{T}}^*(LX) \quad \text{as } K[u] \text{ modules.}$$

These isomorphisms throw the fundamental exact sequence of cyclic homology onto the one for equivariant cohomology.

Proof. This is a simple consequence of the definition of formal, 2.1 and Theorem A.

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