University of California, Berkeley
Fall 2021, Math 215A
Midterm 2 **SOLUTIONS**

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GSI: Ethan Dlugie

November 1 to 5, 2021

Instructions

You may work on this exam at any point between 9:00AM on November 1, 2021 and 11:59PM on November 5, 2021. Before the November 5 deadline, complete and upload your solutions to the three problems below on Gradescope.

You are permitted to use your textbook, your notes, and any other resources that you have produced or that the instructor has provided as part of this course. No external material is allowed.

You are not permitted to discuss these problems with your fellow classmates or anyone else until the solutions have been posted. All work must be your own.

Email the instructor (Professor Nadler) or GSI (Ethan Dlugie) if you have any issues. Otherwise, good luck!

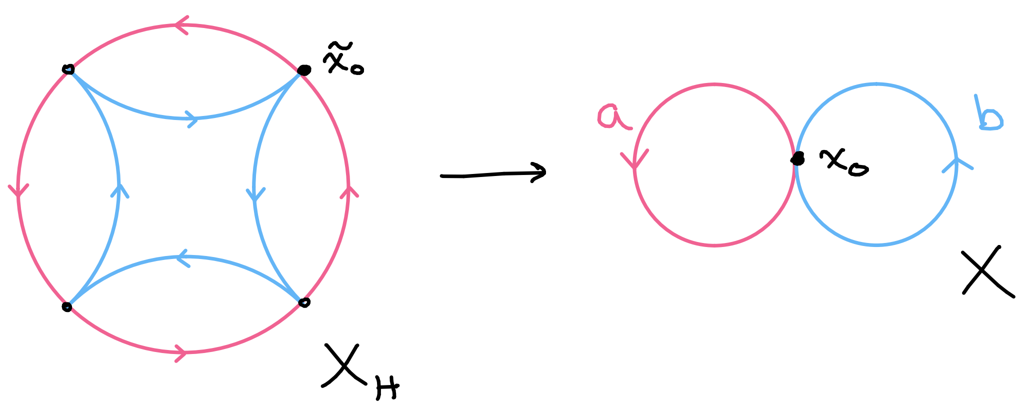
Questions

1. Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Give an explicit example of a finite index, normal subgroup H of F_2 that does not contain the element ab^2. (By "explicit", we mean give a generating set for H.) Of course, you should prove that your H satisfies the requirements.

   Solution: Subgroups of F_2 can be constructed via covering spaces of $X = S_1 \vee S_1$. Orient the two loops and label them a and b to fix an isomorphism $\pi_1(X) \approx F_2$. For any subgroup $H \leq \pi_1(X)$, we have

   - $ab^2 \in H$ if and only if the loop ab^2 in X lifts to a loop in the associated covering space X_H (Proposition 1.31),
   - H is finite index in F_2 if and only if X_H, thought of as a graph, has finitely many vertices (Proposition 1.32), and
   - H is a normal subgroup of F_2 if and only if the deck transformation group acts transitively on the vertices of X_H (Proposition 1.39).

   The *path* ab^2 (rather than the loop), always has a lift to X_H. So we seek a covering space of X where the lift of ab^2 does not close up to form a loop. Consider the covering space X_H as shown here:

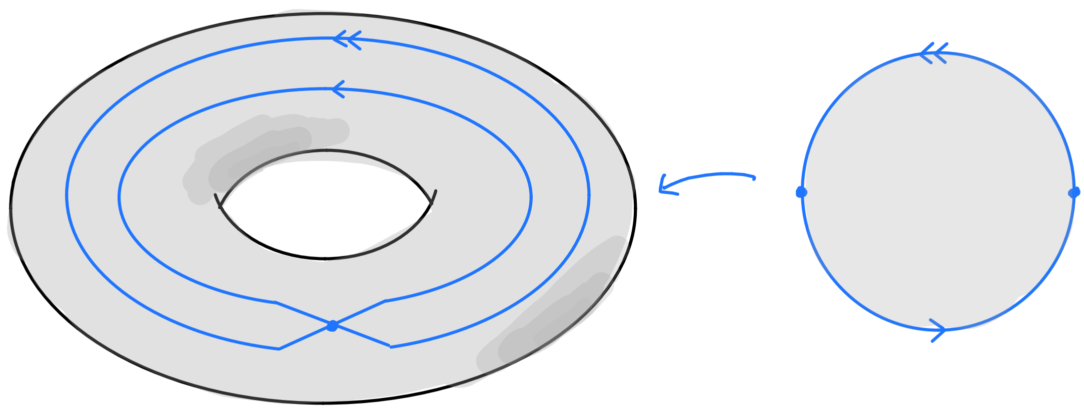


   The lift of ab^2 is unique (Proposition 1.34), and here it is easy to see that it does not form a closed loop. So $\pi_1(X_H, \tilde{x}_0) \approx H \leq F_2$ does not contain ab^2. By counting the vertices of X_H, we easily see that H has index 4 in F_2. And rotations of X_H are clearly deck transformations and act transitively on the 4 vertices, so H is normal. The fundamental group of X_H is free on five generators, and it embeds in F_2 as the subgroup

$$H = \langle ab, a^2b^2, a^3b^3, a^4, b^4 \rangle.$$

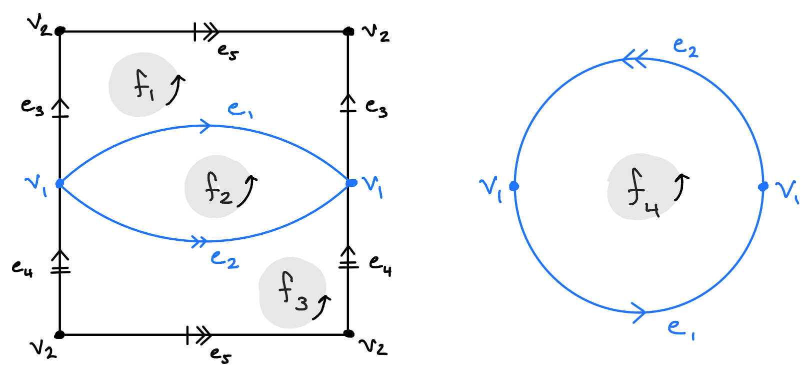
∎

2. Consider a space Z constructed by gluing the boundary of a disk along the curve shown in the torus:



Compute the homology groups of Z.

Solution: The resulting space Z can be given the structure of a 2-complex with two 0-cells v_1, v_2, five 1-cells $e_1, \ldots, e_5$, and four 2-cells $f_1, \ldots, f_4$ as indicated in this diagram:



So we have a cellular chain complex

$$C_\bullet^{CW}(Z) = \{\cdots \to 0 \to \mathbb{Z}\langle f_1, f_2, f_3, f_4\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1, e_2, e_3, e_4, e_5\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v_1, v_2\rangle\}$$

$$\partial_2 = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \qquad \partial_1 = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

This gives 0-th homology

$$H_0(X) = \langle v_1, v_2 \mid v_1 - v_2 \rangle = \langle v_1 \rangle \approx \mathbb{Z},$$

first homology

$$\begin{aligned} H_1(X) &= \langle e_1, e_2, e_3 + e_4, e_5 \mid e_1 - e_5, -e_1 + e_2, -e_2 + e_5, e_1 + e_2 \rangle \\ &= \langle e_1, e_2, e_3 + e_4 \mid -e_1 + e_2, -e_2 + e_1, e_1 + e_2 \rangle \\ &= \langle e_1, e_3 + e_4 \mid 2e_1 \rangle \\ &\approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}, \end{aligned}$$

and second homology

$$H_2(X) = \langle f_1 + f_2 + f_3 \rangle \approx \mathbb{Z}.$$

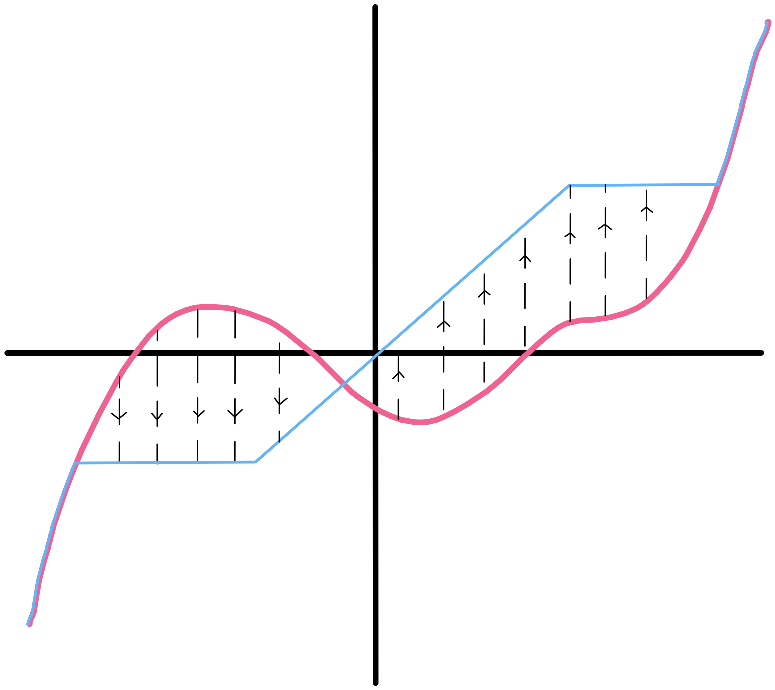
Since Z is a 2-complex, all $H_n(Z)$ are zero for $n \geq 3$.

3. Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial with real coefficients. Show that p can always be extended to a continuous map of one-point compactifications $\hat{p} : S^1 \to S^1$. Find a formula for the degree of $\hat{p}$ in terms of the polynomial p.

Solution: If p is a constant polynomial, then we just define $\hat{p} : S^1 \to S^1$ to be the corresponding constant map. If p is non-constant, then we have $\lim_{|x|\to\infty} |p(x)| = \infty$. So we may set $\hat{p}(\infty) = \infty$ to get a continuous extension to $S^1 = \hat{\mathbb{R}}$.

Suppose now that p is an even degree polynomial. (This includes constant polynomials.) Then p has a maximum or minimum value. In particular, p is not surjective. So $\hat{p} : S^1 \to S^1$ is not surjective, and we get $\deg \hat{p} = 0$.

Suppose that p has odd degree. We can take a compact interval outside of which p has no zeroes and homotope p inside this interval so that it has a unique zero at $x = 0$ and so that $p(x) = \pm x$ around zero. The figure below gives the spirit of the construction. Evidently, this extends to a homotopy of $\hat{p}$. But then the degree of $\hat{p}$ is just the local degree of p at $x = 0$. This value is ± 1, according to the sign of $p(x) = \pm x$. This sign is the same as that of the leading coefficient of p.



In summary, for a general polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ we have

$$\deg \hat{p} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \operatorname{sgn}(a_n) & \text{if } n \text{ is odd.} \end{cases}$$

If we want a closed-form expression, we can write

$$\deg \hat{p} = \operatorname{sgn}(a_n) \left(\frac{1 + (-1)^n}{2} \right).$$

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