Let $E$ be an elliptic curve, i.e. a degree 3 curve embedded in $\mathbb{P}^2$. Show that $\text{Pic}(E) = E \times \mathbb{Z}$.

1 Definitions

I’ll redefine and restate everything, just so everyone is on the same page.

**Definition 1.1.** An *elliptic curve* is a degree 3 curve embedded in $\mathbb{P}^2$ along with a distinguished point $P_0$. Equivalently (as shown in class), it is a curve with geometric genus 1, i.e. its sheaf of top differential forms has global sections of dimension one.

**Definition 1.2.** A *Weil divisor* on a nonsingular projective curve $C$ is a (finite) formal sum of closed points with $\mathbb{Z}$-coefficients. The set of Weil divisors is $\text{Div}^p C$. A *principal Weil divisor* is one which arises as the zeroes and poles of a meromorphic function. The *class group* is the Weil divisors modulo the principal Weil divisors, $\text{Cl}^p C$. A *line bundle* is a locally free sheaf of rank one. The *Picard group* $\text{Pic}^p C$ is the group of isomorphism classes of line bundles with the tensor product as multiplication and dual as inverse.

**Theorem 1.1.** $\text{Pic}^p C \cong \text{Cl}^p C$.

**Theorem 1.2** (Bezout’s theorem). Let $C, D$ be two nonsingular curves in $\mathbb{P}^2$ of degree $c$ and $d$. Then the sum of the intersection multiplicities is $cd$.

2 Solution

To compute the Picard group, then, we should try to determine the rational functions. Some of these results can be generalized.

**Proposition 2.1.** Let $E$ be an elliptic curve. Every principal divisor has degree 0.

**Proof.** Let $f$ be the homogeneous function in $k[x, y, z]$ that cuts out $E$. Nonsingular implies integral, so $f$ is irreducible. The rational functions are $K_e \subset K(\mathbb{P}^2)$. One can check that $K(\mathbb{P}^2)$ consists exactly of degree zero rational functions in $\mathbb{C}[x, y, z]$ and $f$ is homogeneous, so the proposition follows.

**Corollary 1.** Then, the degree map $\deg : \text{Cl}(E) \to \mathbb{Z}$ is well-defined and surjective. Denote its kernel by $\text{Cl}^0(E)$. Then $\text{Cl}(E) = \text{Cl}^0(E) \times \mathbb{Z}$.

**Proof.** All principal divisors have degree zero, so it is well-defined, and it is surjective because we can sum points as we please. The second result follows since $\mathbb{Z}$ is free, and in particular projective, so the degree map splits.

**Theorem 2.1.** $\text{Cl}^0(E) \cong E$ (as sets)

**Proof.** For now, let $P_0$ be some point to be determined later. Consider the map $E \to \text{Cl}^0(E)$ that sends $P \mapsto P - P_0$.

**Injectivity.** We generalize an argument from earlier. Every principal Weil divisor on $E$ can be obtained from some $f/g \in K(\mathbb{P}^2)$, as long neither $f, g$ are multiples of the defining equation for $E$. Using Bezout’s theorem, we see that the resulting principal Weil divisor on $E$ must have effective divisors in degree a multiple of three, and negative divisors in a degree a multiple of three. In particular, $P - Q$ cannot be a principal divisor. Thus, two Weil divisors $P - P_0$ and $Q - Q_0$ cannot be equivalent in $\text{Cl}(E)$.

**Surjectivity.** We will use the following idea. On $\mathbb{P}^2$, any two hypersurfaces of the same degree, as Weil divisors, are equivalent in the class group. Thus, the same is true if we intersect with $E$. In particular, it is true for lines. We will intersect various lines in $\text{Div}(\mathbb{P}^2)$ with $E$ to get various triple-sums of points in $\text{Div}(E)$, which must all be equivalent in $\text{Cl}(E)$.

We want to choose a point $P_0$ which is an inflection point under a given embedding, thereby making $3P_0$ such a linear equivalence class as discussed above. To do this, we can restrict to any open coordinate chart (i.e. $k^2$) and find a zero for the Hessian for the resulting plane curve, giving us some point $P_0$. 

Given two points \( P, Q \) on a curve \( C \), there is a line passing through them. By Bezout’s theorem, the line intersects \( C \) and another point \( R \) (not necessarily distinct). Thus, \( P + Q + R = 3P_0 \). We can repeat this to get that \( P_0 + Q + T = 3P_0 \), so \( P + Q + R = P_0 + Q + T \), i.e. \( P + R = T + P_0 \), i.e. \( (P - P_0) + (R - P_0) = (T - P_0) \). Note that this process can be done in reverse, i.e. given \( R \) and \( T \), we can recover \( P \), and therefore we can also simplify expressions of the form \( (T - P_0) - (R - P_0) = P - P_0 \).

So given some degree zero divisor \( \sum_{i=1}^{\ell} n_i P_i \), since the degree is zero we can rewrite this \( \sum_{i=1}^{\ell} n_i (P_i - P_0) \). Using the above method, we can reduce this expression to the form \( P - P_0 \), so the map is surjective.

**Corollary 2.** Combining the previous results, we see that \( \text{Pic}(E) = E \times \mathbb{Z} \), where the group structure on \( E \) is given as in the proof above.