Let E be an elliptic curve, i.e. a degree 3 curve embedded in $\mathbb{P}^2$. Show that $\operatorname{Pic}(E) = E \times \mathbb{Z}$.

1 Definitions

I'll redefine and restate everything, just so everyone is on the same page.

Definition 1.1. An *elliptic curve* is a degree 3 curve embedded in $\mathbb{P}^2$ along with a distinguished point P_0. Equivalently (as shown in class), it as a curve with geometric genus 1, i.e. its sheaf of top differential forms has global sections of dimension one.

Definition 1.2. A *Weil divisor* on a nonsingular projective curve C is a (finite) formal sum of closed points with $\mathbb{Z}$-coefficients. The set of Weil divisors is $\operatorname{Div}(C)$. A *principal Weil divisor* is one which arises as the zeroes and poles of a meromorphic function. The *class group* is the Weil divisors modulo the principal Weil divisors, $\operatorname{Cl}(C)$. A *line bundle* is a locally free sheaf of rank one. The *Picard group* $\operatorname{Pic}(C)$ is the group of isomorphism classes of line bundles with the tensor product as multiplication and dual as inverse.

Theorem 1.1. $\operatorname{Pic}(C) \cong \operatorname{Cl}(C)$.

Theorem 1.2 (Bezout's theorem)**.** *Let C, D be two nonsingular curves in $\mathbb{P}^2$ of degree c and d. Then the sum of the intersection multiplicities is cd.*

2 Solution

To compute the Picard group, then, we should try to determine the rational functions. Some of these results can be generalized.

Proposition 2.1. *Let E be an elliptic curve. Every principal divisor has degree 0.*

Proof. Let f be the homogeneous function in $k[x, y, z]$ that cuts out E. Nonsingular implies integral, so f is irreducible. The rational fuctions are $K(E) \subset K(\mathbb{P}^2)$. One can check that $K(\mathbb{P}^2)$ consists exactly of degree zero rational functions in $\mathbb{C}(x, y, z)$ and f is homogeneous, so the proposition follows. □

Corollary 1. *Then, the degree map* $\deg : \operatorname{Cl}(E) \to \mathbb{Z}$ *is well-defined and surjective. Denote its kernel by* $\operatorname{Cl}^0(E)$*. Then* $\operatorname{Cl}(E) = \operatorname{Cl}^0(E) \times \mathbb{Z}$*.*

Proof. All principal divisors have degree zero, so it is well-defined, and it is surjective because we can sum points as we please. The second result follows since $\mathbb{Z}$ is free, and in particular projective, so the degree map splits. □

Theorem 2.1. $\operatorname{Cl}^0(E) \cong E$ *(as sets)*

Proof. For now, let P_0 be some point to be determined later. Consider the map $E \to \operatorname{Cl}^0(E)$ that sends $P \mapsto P - P_0$.

Injectivity. We generalize an argument from earlier. Every principal Weil divisor on E can be obtained from some $f/g \in K(\mathbb{P}^2)$, as long neither f, g are multiples of the defining equation for E. Using Bezoit's theorem, we see that the resulting principal Weil divisor on E must have effective divisors in degree a multiple of three, and negative divisors in a degree a multiple of three. In particular, $P - Q$ cannot be a principal divisor. Thus, two Weil divisors $P - P_0$ and $Q - Q_0$ cannot be equivalent in $\operatorname{Cl}(E)$.

Surjectivity. We will use the following idea. On $\mathbb{P}^2$, any two hypersurfaces of the same degree, as Weil divisors, are equivalent in the class group. Thus, the same is true if we interesect with E. In particular, it is true for lines. We will intersect various lines in $\operatorname{Div}(\mathbb{P}^2)$ with E to get various triple-sums of points in $\operatorname{Div}(E)$, which must all be equivalent in $\operatorname{Cl}(E)$.

We want to choose a point P_0 which is an inflection point under a given embedding, thereby making $3P_0$ such a linear equivalence class as discussed above. To do this, we can restrict to any open coordinate chart (i.e. $\mathbb{A}^2$) and find a zero for the Hessian for the resulting plane curve, giving us some point P_0.

Given two points P, Q on a curve C, there is a line passing through them. By Bezout's theorem, the line intersects C and another point R (not necessarily distinct. Thus, $P + Q + R = 3P_0$. We can repeat this to get that $P_0 + Q + T = 3P_0$, so $P + Q + R = P_0 + Q + T$, i.e. $P + R = T + P_0$, i.e. $(P - P_0) + (R - P_0) = (T - P_0)$. Note that this process can be done in reverse, i.e. given R and T, we can recover P, and therefore we can also simplfy expressions of the form $(T - P_0) - (R - P_0) = P - P_0$.

So given some degree zero divisor $\sum_{n=1}^{\ell} n_i P_i$, since the degree is zero we can rewrite this $\sum_{n=1}^{\ell} n_i (P_i - P_0)$. Using the above method, we can reduce this expression to the form $P - P_0$, so the map is surjective. □

Corollary 2. *Combining the previous results, we see that* $\mathrm{Pic}(E) = E \times \mathbb{Z}$, *where the group structure on* E *is given as in the proof above.*