The Geometric Nature of the Fundamental Lemma

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Teleportation was eighth.
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To understand **functoriality**, we must study **Langlands reciprocity**...
Linearization of subspaces

Let $X$ be a measure space.

Subspaces $Y \subset X \mapsto$ integral distributions $Y(\varphi) = \int_Y \varphi$

Now can add $Y_1 + Y_2$ and scale $cY$ subspaces.

Suppose symmetry group $G$ acts on $X$ preserving measure.

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Now consider \textit{group} $G$ with a conjugation invariant measure.

Importance of linearization of conjugacy classes:

characters of $G$-representations $\rightsquigarrow$ $G$-invariant distributions

Given $G$-representation $V$, can form distributional character:

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\chi_V(\varphi) = \int_G \varphi(g) \text{Tr}_V(g) \, dg
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Example: Finite groups

Specialize to finite group $G$. Then

$$G$$-invariant distributions $= \text{class functions}

\textbf{Theorem}

Characters $\chi_V$ of irreducible $G$-representations $V$ form basis for class functions. Rescaled characters $\hat{\chi}_V = \chi_V / \dim V$ idempotents with respect to convolution $\hat{\chi}_V * \hat{\chi}_V = \hat{\chi}_V$.

\textbf{Interpretation}

Class functions $= \text{functions on space of irreducible representations.}$
Rescaled characters $\hat{\chi}_V$ are characteristic functions of points.
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It is difficult to construct representations.

We have trivial representation $\text{Tr}$ and method of induction.

Given group $G$, and subgroup $\Gamma \subset G$, form “unitary induction”

$$\text{Ind}_{\Gamma}^G(\text{Tr}) = L^2(G/\Gamma)$$

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Frobenius Character Formula

$G$ finite group, $\Gamma \subset G$ subgroup.

Character of induced representation $L^2(G/\Gamma)$

$$\chi^G_\Gamma(\varphi) = \sum_{\gamma \in \Gamma/\Gamma} a_\gamma O_\gamma(\varphi)$$

Volumes of quotients of centralizers

$$a_\gamma = |\Gamma_\gamma \backslash G_\gamma|$$

Integrals over conjugacy classes

$$O_\gamma(\varphi) = \int_{[\gamma]} \varphi = \sum_{x \in G_\gamma \backslash G} \varphi(x^{-1} \gamma x)$$

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**Frobenius Character Formula**

Let $G$ be a finite group, $\Gamma \subset G$ a subgroup. The character of the induced representation $L^2(G/\Gamma)$ is given by

$$\chi_G^\Gamma(\varphi) = \sum_{\gamma \in \Gamma/\Gamma} a_\gamma O_\gamma(\varphi)$$

where $a_\gamma = |\Gamma \gamma \backslash G_\gamma|$ are the volumes of the quotients of centralizers. The integrals over conjugacy classes are

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The image contains a portrait of Ferdinand Georg Frobenius, 1849–1917.
Poisson Summation Formula

**R** additive group, **Z** ⊂ **R** discrete subgroup.

Character of induced representation \( L^2(\mathbb{R}/\mathbb{Z}) \)

\[
\chi_{\mathbb{Z}}^\mathbb{R}(\varphi) = \sum_{n \in \mathbb{Z}} \varphi(n)
\]

(Fourier analysis provides isomorphism

\[
L^2(\mathbb{R}/\mathbb{Z}) \simeq \bigoplus_{\lambda \in \mathbb{Z}} \mathbb{C}\langle e^{2\pi i \lambda} \rangle
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Hence identification of characters

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\sum_{n \in \mathbb{Z}} \varphi(n) = \sum_{\lambda \in \mathbb{Z}} \hat{\varphi}(\lambda).
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Siméon Denis Poisson
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Arthur-Selberg Trace Formula

\( \mathbb{G} \) reductive algebraic group over number field \( F \).
Think \( \mathbb{G} = GL(n) \) and \( F = \mathbb{Q} \).

\( \mathbb{A}_F \) adèles of \( F \) of all hypothetical “Laurent series expansions” of elements in the form of \( p \)-adic and real numbers.

Then \( G = \mathbb{G}(\mathbb{A}_F) \) is a locally compact group, and \( \Gamma = \mathbb{G}(F) \subset \mathbb{G}(\mathbb{A}_F) \) is a discrete subgroup.

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\chi^G_G(\varphi) = \sum_{\gamma \in \Gamma/\Gamma} a_{\gamma} \mathcal{O}_{\gamma}(\varphi) + \cdots
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Upshot: character involves integrals over conjugacy classes in real and \( p \)-adic Lie groups.
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For simplicity, let’s consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

Three types of orbits under conjugation:
- hyperbolic: $\det < 0$.
- nilpotent: $\det = 0$.
- elliptic: $\det > 0$.

We will focus on the two elliptic orbits $\mathcal{O}_{A_+}, \mathcal{O}_{A_-} \subset \mathfrak{s}(2, \mathbb{R})$ through the matrices

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A_+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_- = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
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are both conjugate to the matrix

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One says that $A_+$ and $A_-$ are stably conjugate.
Stable conjugacy

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Invariant polynomials

Orbits of $SL(2, \mathbb{R})$ acting on its Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \cong \mathbb{R}^3$.

Said another way, the two elliptic orbits $\mathcal{O}_{A+}, \mathcal{O}_{A-} \subset \mathfrak{s}(2, \mathbb{R})$ coalesce into a single conjugacy class $\mathcal{O}_A \subset \mathfrak{s}(2, \mathbb{C})$ cut out by the invariant polynomial $\det = 1$.
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Linearization of adjoint orbits

Orbits of $SL(2, \mathbb{R})$ acting on its Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \cong \mathbb{R}^3$.

Consider the distributions given by integrating over the elliptic orbits

$$O_{A^+}(\varphi) = \int_{O_{A^+}} \varphi \quad O_{A^-}(\varphi) = \int_{O_{A^-}} \varphi$$

with respect to an invariant measure.
The distributions $\mathcal{O}_{A+}, \mathcal{O}_{A-}$ span a two-dimensional complex vector space. It admits the alternative basis

$$\mathcal{O}_{st} = \mathcal{O}_{A+} + \mathcal{O}_{A-}$$
$$\mathcal{O}_{tw} = \mathcal{O}_{A+} - \mathcal{O}_{A-}$$

Here $st$ stands for stable and $tw$ stands for twisted.

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Algebraic variety defined by invariant polynomial

\[ \det = 1. \]

Stable distribution is object of

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Twisted distributions

What to do with twisted distribution

\[ \mathcal{O}_{tw} = \mathcal{O}_{A_+} - \mathcal{O}_{A_-} \]

Distinguishes between \( \mathcal{O}_{A_+} \) and \( \mathcal{O}_{A_-} \)

though no invariant polynomial separates them.

Twisted distribution appears to be noncanonical: exchanging terms

\[ \mathcal{O}_{A_+} \leftrightarrow \mathcal{O}_{A_-} \]

induces sign change

\[ \mathcal{O}_{tw} \leftrightarrow -\mathcal{O}_{tw} \]
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Interpretation via Fourier analysis

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Alternative basis

$$\mathcal{O}_{st} = \mathcal{O}_{A+} + \mathcal{O}_{A-}$$

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results from *Fourier analysis* on set of orbits

$$\{ \mathcal{O}_{A+}, \mathcal{O}_{A-} \}$$
What is the Fundamental Lemma all about?

Basic idea

Langlands's theory of endoscopy, and the Fundamental Lemma at its heart, confirms that one can systematically express twisted distributions in terms of stable distributions or nonconstant Fourier modes in terms of constant Fourier modes.
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*twisted distributions in terms of stable distributions*

*nonconstant Fourier modes in terms of constant Fourier modes*
Example continued

Endoscopy relates twisted distribution

\[ \mathcal{O}_{tw} = \mathcal{O}_{A+} - \mathcal{O}_{A-} \]

to stable distribution on Lie algebra \( \mathfrak{so}(2, \mathbb{R}) \) \( \cong \mathbb{R} \) of subgroup \( \text{SO}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{R}) \)

stabilizing matrices

\[ A_+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A_- = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

Outside of bookkeeping, this is empty of content since \( \text{SO}(2, \mathbb{R}) \) is abelian, and so its orbits in \( \mathfrak{so}(2, \mathbb{R}) \) are single points.
Example continued

**Endoscopy** relates twisted distribution

\[ O_{tw} = O_{A+} - O_{A-} \]

to stable distribution on Lie algebra \( \mathfrak{so}(2, \mathbb{R}) \cong \mathbb{R} \) of subgroup \( \text{SO}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{R}) \)

stabilizing matrices

\[
A_+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A_- = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

Outside of bookkeeping, this is empty of content since \( \text{SO}(2, \mathbb{R}) \) is abelian, and so its orbits in \( \mathfrak{so}(2, \mathbb{R}) \) are single points.
Why is the Fundamental Lemma difficult?

General theory of endoscopy is deep and elaborate.

Key challenge

Extraordinary difficulty of the Fundamental Lemma, and also its mystical power, emanates from fact that sought-after stable distributions live on so-called endoscopic groups $H$ with little apparent geometric relation to original group $G$. 
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Endoscopic groups

To find relation between group $G$ and endoscopic group $H$, one must pass to Langlands dual groups

“noncommutative Pontryagin dual group” of “geometric characters”

There one finds $H^\vee$ is naturally subgroup of $G^\vee$.

Example
Consider the symplectic group $G = Sp(2n)$.
The special orthogonal group $H = SO(2n)$ is not a subgroup.
But $H^\vee = SO(2n)$ is a subgroup of $G^\vee = SO(2n + 1)$.

Endoscopy gives precise relationship

twisted distributions on $Sp(2n) \rightsquigarrow$ stable distributions on $SO(2n)$
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**twisted distributions on $Sp(2n)$ $\rightsquigarrow$ stable distributions on $SO(2n)$**
Low rank example

Foreground: roots of the group \( G = \text{Sp}(4) \) with roots of the endoscopic group \( H = \text{SO}(4) \) highlighted.

Background: roots of the Langlands dual group \( G^\vee = \text{SO}(5) \) with roots of the subgroup \( H^\vee = \text{SO}(4) \) highlighted.
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Real versus $p$-adic Lie groups

Detailed conjectures organizing the intricacies of the transfer of distributions first appear in Langlands’s joint work with Shelstad.

General setting needed for applications to number theory and harmonic analysis: $p$-adic and real Lie groups (algebraic groups over local fields).

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What became known as the Fundamental Lemma is the most basic conjecture for $p$-adic groups.

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From $p$-adic groups to loop groups

Dictionary between arithmetic and geometry.

One-dimensional Arithmetic
- Number field $F$
- Rational numbers $\mathbb{Q}$
- $p$-adic field
- $p$-adic group

One-dimensional Geometry
- Smooth projective curve $X$
- Projective line $\mathbb{P}^1$
- Formal disk $D$
- Loop group $LG$

Theorem (Waldspurger)
To prove Fundamental Lemma, it suffices to prove its analogue in the geometric setting.

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**Loop Grassmannians**

Orbital integrals of Fundamental Lemma in geometric setting are equivalent to counting points in subvarieties of Grassmannians.

**Definition**

Let $LG$ be loop group. Let $L_+G \subset LG$ be subgroup of arcs. Loop Grassmannian $Gr_G$ is homogenous space $LG/L_+G$.

Why Grassmannian? $\infty$/2-dim subspaces of $\infty$-dim vector space.
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Geometric cousin of affine building.
Affine Springer fibers

Now the subvarieties...

Definition

Let $\xi$ be element of Lie algebra of $LG$.

Affine Springer fiber $X_\xi \subset Gr_G$ is fixed-point locus of $\xi$. 
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Example for $\xi$ diagonal with distinct eigenvalues.
Basic structure of affine Springer fibers

- $X_\xi$ is finite-dimensional increasing union of projective varieties.
- $X_\xi/\Lambda_\xi$ quotient by symmetry lattice is projective variety.
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From point counts to cohomology

Trace formula: count points in algebraic variety by calculating traces of Galois symmetries acting on topological cohomology.

Grothendieck 1928–

Lefschetz 1884–1972

Now can stand on the shoulders of giants: Kazhdan-Lusztig, Goresky-MacPherson, Beilinson-Bernstein-Deligne-Gabber,…

Challenge: cohomology of affine Springer fibers quantifiably too complicated to calculate in any combinatorially explicit form.
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Trivial case: when vector field is generic, for example sum of linearly independent commuting vector fields.

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Compactified Jacobians

Beautiful insight

Affine Springer fibers modulo natural symmetries parametrize generalized line bundles on curves.

Deformations to simpler curves provide deformations to simpler affine Springer fibers.

Striking consequence

Fundamental Lemma for unitary groups!
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**Hitchin fibration**

X smooth projective curve (Riemann surface).

*Hitchin moduli* $\mathcal{M}_G(X)$ parametrizes $G$-bundle on $X$ together with twisted endomorphism.

*Base* $\mathcal{A}_G(X)$ parametrizes possible eigenvalues of twisted endomorphism (spectral curve).

*Integrable system* $\mathcal{M}_G(X) \to \mathcal{A}_G(X)$ assigns characteristic polynomial of endomorphism.

Fibers parametrize generalized line bundles on spectral curves.

*Hitchin fibration* organizes deformations of affine Springer fibers into a proper finite-dimensional algebraic family.

Nigel Hitchin 1946–
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**Fundamental Lemma** involves distributions on conjugacy classes

adjoint quotient $G/G$

Hitchin moduli space parametrizes twisted maps

curve $X \mapsto$ adjoint quotient $g/G$

Furthermore, stable conjugacy classes involve invariant polynomials

adjoint quotient $G/G \mapsto$ possible eigenvalues $T/W$

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Ngô’s Support Theorem

Main new technical input to proof of Fundamental Lemma. Precise description of the cohomology of the fibers of an integrable system in terms of its generic fibers.

Toy model: consider a family of irreducible curves

\[ f : M \to S, \text{ with } M \text{ and } S \text{ smooth.} \]

Over a Zariski open locus \( S^0 \subset S \), the fibers

\[ M_s = f^{-1}(s), \quad s \in S^0 \]

are topologically equivalent curves, hence their cohomologies \( H^*(M_s) \) form a local system of vector spaces

\[ \mathcal{H} \to S^0 \]

Exercise: the cohomology of any fiber can be recovered from \( \mathcal{H} \).
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Family of plane cubics

\[ y^2 = x^3 + ax + b \]

singular at \((a, b) = (0, 0)\)
Thank you for listening!