11/15/18 Review:

Thm: G fin. gp., V_i irred. rep, $i \in \Phi(G)$, χ_i irred. char of V_i.

Given (V,ρ) rep. of G, define lin. map $P_i : V \to V$ s.t. $P_i(v) = \rho(\chi_i) v$ (This doesn't really make sense, but don't worry)

$$P_i(v) = \rho\Big(\sum_{g\in G} \overline{\chi_i(g)}\,\delta_g\Big)\cdot v := \sum_{g \in G} \overline{\chi_i(g)}\,\rho(g)\cdot v.$$

Then 1) P_i is a map of G-reps (proved last time) i.e. $P_i(\rho(g)\cdot v) = \rho(g) P_i(v)$

2) If $V \simeq \bigoplus_{i\in\Phi(G)} V_i^{\oplus d_i}$, then $\mathrm{Image}(P_i) = V_i^{\oplus d_i}$, and finally

3) $P_i^2 = \frac{|G|}{\dim V_i} P_i$. <u>Pf of 2)</u> Suppose $V = V_j$ irred. Consider $P_i : V_j \to V_j$

Want: $\mathrm{Image}\, P_i = \begin{cases} \langle 0 \rangle & \text{if } i \neq j \\ V_j & \text{if } i = j \end{cases}$. By 1), P_i is a map of G-reps. So by

Schur's Lemma, $P_i = c\,\mathrm{Id}_{V_j}$. Claim: $\mathrm{Tr}(P_i) = (\text{const})\,\langle \chi_i, \chi_j\rangle$

<u>Proof</u>: $\mathrm{Tr}_{V_j}(P_i) = \mathrm{Tr}_{V_j}\Big(\sum_{g\in G} \overline{\chi_i(g)}\,\rho(g)\Big) = \sum_{g \in G} \overline{\chi_i(g)}\, \mathrm{Tr}_{V_j}(\rho(g))$ (By linearity of trace)

$= \sum_{g\in G} \overline{\chi_i(g)}\,\chi_j(g) = |G| \langle \chi_i, \chi_j \rangle$, so const $= |G|$. ∎

Consequence: $\mathrm{Tr}(P_i) = \dim(V_j)\cdot c = |G|\cdot k$ where $k = \begin{cases} 0 & i \neq j \\ 1 & i = j\end{cases}$. Then if $i \neq j$, $\mathrm{Image} = \langle 0 \rangle$; $i = j$, $\mathrm{Image} = V_j$

Now consider general (V,ρ). Claim: P_i is ~~functorial~~ natural in the sense that

$$\begin{array}{ccc} (W,\sigma) & \xrightarrow{\varphi} & (V,\rho) \\ {\scriptstyle P_i}\downarrow & \circlearrowleft & \downarrow{\scriptstyle P_i} \\ (W,\sigma) & \xrightarrow{\varphi} & (V,\rho) \end{array}$$

i.e. P_i is an endomorphism of the identity functor. ☆ Check. ☆

Back to (V,ρ). Suppose $V \simeq \bigoplus_{j\in\Phi(G)} V_j^{\oplus d_j}$

Define $W_j = V_j^{\oplus d_j}$. $\varphi_j^{\ell} : V_j \hookrightarrow V$, $\ell = 1, \dots, d_j$. Apply naturality to φ_j^{ℓ}.

$$\begin{array}{ccc} V_j & \xrightarrow{\varphi_j^{\ell}} & V \\ {\scriptstyle P_i}\downarrow & \circlearrowleft & \downarrow{\scriptstyle P_i} \\ V_j & \xrightarrow{\varphi_j^{\ell}} & V \end{array}$$

Exercise: Thm 2) is proved. (???)

Rings and Modules

<u>Def</u>: A ring R is a set $|R|$ with the operations $+$ (addition) and $\cdot$ (multiplication) satisfying: 1) $(R,+)$ is an abelian group (w/ id 0) 2) $(R,\cdot)$ is a monoid, w/ unit 1. and

3) $+$ and $\cdot$ distribute how you think they should. <u>Ex.</u> 1) $\mathbb{Z}$ is a commutative ring 2) $\mathbb{C}[x_1, \dots, x_n]$ is a commutative ring 3) $\mathbb{Z}/n$ is a commutative ring

4) $M_n(\mathbb{C}) = \langle n\times n$ complex matrices$\rangle$ is a non-commutative ring ($n \neq 1$)

<u>Terminology</u>: If K is a field, then a K-algebra A is a ring with a ring map $c : K \to A$ (a ring map is a set map satisfying $c(k_1+k_2) = c(k_1) + c(k_2)$ and $c(k_1 k_2) = c(k_1)c(k_2)$, $c(0) = 0$, $c(1) = 1$) <u>Exer 1)</u> c is injective unless $A = \langle 0 \rangle$ (hint: kernel of a map is an ideal) 2) Equivalently, A is a K-vector space w.r.t. $+$ and $\cdot$ is K-linear | <u>Key Example</u> $A = \mathbb{C}[G]$, "group algebra of G" (G fin.-dim). $+$ = usual addition, $\cdot$ = <u>convolution</u>, unit δ_e, $f_1 \cdot f_2$ is given by

$$f_1 \cdot f_2(x) = \Big(\Big(\sum_{g\in G} f_1(g)\delta_g\Big) \cdot \Big(\sum_{g\in G} f_2(g)\delta_g\Big)\Big)(x) = \sum_{g_1 g_2 = x} f_1(g_1) f_2(g_2)$$

In other words, $f_1 \cdot f_2 = \sum_{x\in G}\Big(\sum_{g_1 g_2 = x} f_1(g_1) f_2(g_2)\Big)\delta_x$. Special case/Example:

$f_1 = \delta_{g_1}$, $f_2 = \delta_{g_2}$. Then $f_1 \cdot f_2 = \delta_{g_1} \cdot \delta_{g_2} = \sum_{x \in G}(\delta_{g_1}\delta_{g_2})\delta_x = \delta_{g_1 g_2}$

<u>Exercise</u>: $\mathbb{C}[G]$ is comm. $\iff$ G is abelian.

<u>Def</u>: Given a ring R, an R-module M is a set with structures 1) $M,+$ abelian gp. 2). Action map $R \times M \to M$ is assoc., unital, 3) Action distributes over $+$

<u>Ex</u>: 0) R = field K, R-module is a K-vector space. 1) $R = \mathbb{Z} \Rightarrow R$-modules are abelian groups. (Reformulation of R-modules: $R \xrightarrow{\text{Ring map}} \mathrm{End}_{\mathrm{AbGrp}}(M)$) 2) $R = \mathbb{C}[x_1, \dots, x_n] \Rightarrow R$-modules are $\mathbb{C}$-vector spaces equipped with n commuting endomorphisms $x_i : M \to M$ (This encompasses all of Algebraic Geometry)

3) <u>Thm</u> $R = M_n(\mathbb{C}) \Rightarrow R$-modules are all direct sums of copies of $\mathbb{C}^n$ (Morita theory)

<u>Exer.</u> $R = \mathbb{C}[G]$ group alg., R-modules = complex G-representations.