

Thm:  $G$  f.m. gp,  $V_i$  irred. rep,  $i \in \mathbb{I}(G)$ ,  $\chi_i$  irred. char of  $V_i$ .  
 Given  $(V, \rho)$  rep. of  $G$ , define lin. map.  $P_i: V \rightarrow V_i$  s.t.  $P_i(v) = \rho(\chi_i)v$  (This doesn't really make sense but don't worry)

$$P_i(v) = \sum_{g \in G} \overline{\chi_i(g)} \rho(g)v$$

Then 1)  $P_i$  is a map of  $G$ -reps (proved last time) i.e.  $P_i(\rho(g)v) = \rho(g)P_i(v)$

2) If  $V = \bigoplus_{i \in \mathbb{I}(G)} V_i$ , then  $\text{Image}(P_i) = V_i$  and finally

3)  $P_i^2 = \frac{|G|}{\dim V_i} P_i$ . Pf of 2) Suppose  $V = V_i$  irred. Consider  $P_i: V \rightarrow V_i$

Want:  $\text{Image } P_i = \begin{cases} \langle 0 \rangle & \text{if } i \neq j \\ V_i & \text{if } i = j \end{cases}$ . By 1),  $P_i$  is a map of  $G$ -reps. So by

Schur's Lemma,  $P_i = c I_{V_i}$ . (Claim:  $\text{Tr}(P_i) = (\text{const}) \cdot \langle \chi_i, \chi_j \rangle$ )

Proof:  $\text{Tr}_{V_j}(P_i) = \text{Tr}_{V_j}(\sum_{g \in G} \overline{\chi_i(g)} \rho(g)) = \sum_{g \in G} \overline{\chi_i(g)} \text{Tr}_{V_j}(\rho(g))$   
(by linearity of trace)

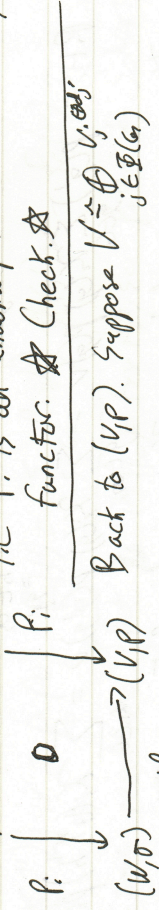
$$= \sum_{g \in G} \chi_i(g) \chi_j(g) = |G| \langle \chi_i, \chi_j \rangle, \text{ so } \text{const} = |G|.$$

Consequence:  $\text{Tr}(P_i) = \dim(V_i) \cdot c = |G| \cdot c$  where

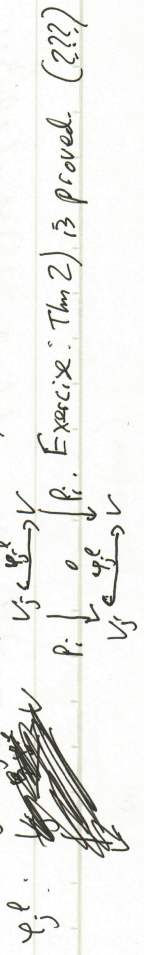
$$c = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}. \text{ Then if } i \neq j, \text{ Image} = \langle 0 \rangle, \text{ if } i = j, \text{ Image} = V_i$$

Now consider general  $(V, \rho)$ . (Claim:  $P_i$  is  $\text{natural}$  functorial in the sense

that  $(W, \sigma) \xrightarrow{\varphi} (V, \rho)$  i.e.  $P_i$  is an endomorphism of the identity functor. Check.  $\star$



Define  $W_j = V_j \otimes \mathbb{C}$ .  $\varphi_j: V_j \otimes \mathbb{C} \rightarrow V_j$ ,  $j = 1, \dots, d_j$ . Apply naturality to



Exercise: Thm 2) is proved. (???)

Rings and Modules

Def: A ring  $R$  is a set  $|R|$  with the operations  $+$  (addition) and  $\cdot$  (multiplication) satisfying: 1)  $(R, +)$  is an abelian group 2)  $(R, \cdot)$  is a monoid w/unit 1. and  $(w/id 0)$

3)  $+$  and  $\cdot$  distribute how you think they should. Ex: 1)  $\mathbb{Z}$  is a commutative ring

2)  $\mathbb{C}[X_1, \dots, X_n]$  is a commutative ring 3)  $\mathbb{Z}/n\mathbb{Z}$  is a commutative ring

4)  $M_n(\mathbb{C}) = \{n \times n \text{ complex matrices}\}$  is a non-commutative ring ( $M \neq 1$ )

Terminology: If  $K$  is a field, then a  $k$ -algebra  $A$  is a ring with a ring map  $c: K \rightarrow A$  (a ring map is a set map satisfying  $c(k_1 + k_2) = c(k_1) + c(k_2)$  and  $c(k_1 k_2) = c(k_1) c(k_2)$ ,  $c(0) = 0, c(1) = 1$ ) Ex: 1)  $C$  is injective unless

$A = \langle 0 \rangle$  (hint: kernel of a map is an ideal) 2) Equivalently,  $A$  is a  $k$ -vector space

w/  $(+)$  and  $\cdot$  is  $k$ -linear | Key Example:  $A = \mathbb{C}[G]$  "group algebra of  $G$ "  
( $G$  fin. dim.).  $+$  = usual addition,  $\cdot$  = convolution, unit  $1_{G}$ .  $f_1 \cdot f_2$  is given by

$$f_1 \cdot f_2(x) = \left( \sum_{g \in G} f_1(g) \delta_g \right) \cdot \left( \sum_{g \in G} f_2(g) \delta_g \right) (x) = \sum_{g_1 g_2 = x} f_1(g_1) f_2(g_2)$$

In other words,  $f_1 \cdot f_2 = \sum_{x \in G} \left( \sum_{g_1 g_2 = x} f_1(g_1) f_2(g_2) \right) \delta_x$ . Special case / Example:

$$f_1 = \delta_g, f_2 = \delta_h. \text{ Then } f_1 \cdot f_2 = \delta_g \cdot \delta_h = \sum_{x \in G} (\delta_g \delta_h) \delta_x = \delta_{gh}$$

Exercise:  $\mathbb{C}[G]$  is comm.  $\Leftrightarrow G$  is abelian.

Def: Given a ring  $R$ , an  $R$ -module  $M$  is a set with structures  $(M, +)$  abelian gp.

2) Action map  $R \times M \rightarrow M$  is assoc, unital, 3) Action distributes over  $+$

Ex: 0)  $R = \text{field } K$ ,  $R$ -module is  $K$ -vector space. 1)  $R = \mathbb{Z} \Rightarrow R$ -modules are abelian groups.

(Reformulation of  $R$ -modules): 2)  $R = \mathbb{C}[X_1, \dots, X_n] \Rightarrow R$ -modules are  $\mathbb{C}$ -vector spaces equipped with  $n$  commuting  $R$ -ring map

endomorphisms  $X_i: M \rightarrow M$  (This encompasses all of Algebraic Geometry)

3)  $\prod_{i=1}^n R = M_n(\mathbb{C}) \Rightarrow R$ -modules are all direct sums of copies of  $\mathbb{C}$  (Morita theory)

Exer:  $R = \mathbb{C}[G]$  group alg.  $R$ -modules = complex  $G$ -representations.