23 November 15, 2018

Theorem 23.1. *Let G be a finite group, V_i be irreducible representations with $i \in \Phi(G)$, and χ_i are irreducible characters of V_i. Given some representation (V, ρ) of G, we define a linear map*

$$P_i : V \to V$$

by

$$P_i(v) = \rho(\chi_i) \cdot v = \rho\left(\sum_{g \in G} \chi_i(g) f_g\right) \cdot v = \sum_{g \in G} \overline{\chi_i(g)} \rho(g) \cdot v.$$

Then

1. *P_i is a map of G-representations $P_i(\rho(g) \cdot v) = \rho(g) P_i(v)$.*
2. *If we write V as a sum of irreducibles*
$$V \cong \bigoplus_i V_i^{\oplus d_i}$$
then we have that the image of P_i is $V_i^{\oplus d_i}$
3. $$P_i^2 = \frac{|G|}{\dim V_i} P_i$$
i.e. it is almost an orthogonal projection.

We proved (1) in the last lecture, but we include the proof here for convenience.

Proof of (1). To be a map of $G - reps$, we must have the following diagram commute for all $g \in G$:

$$\begin{tikzcd}
V \arrow[r, "P_i"] \arrow[d, "\rho(g)"] & V \arrow[d, "\rho(g)"] \\
V \arrow[r, "P_i"] & V
\end{tikzcd}$$

that is

$$P_i \circ \rho(g) = \rho(g) \circ P_i$$

On the left hand size, we have

$$\left(\sum_{g' \in G} \chi_i(g') \rho(g')\right) \circ \rho(g) = \sum_{g' \in G} \chi_i(g') \rho(g' g)$$

We let $h = g'g$ and re-index this as

$$\sum_{h \in G} \chi_i(hg^{-1}) \rho(h)$$

And on the right we have

$$\rho(g) \circ \sum_{g' \in G} \chi_i(g')\rho(g') = \sum_{g' \in G} \chi_i(g')\rho(gg')$$

We let $h = gg'$ and re-index this as

$$\sum_{h \in G} \chi(g^{-1}h)\rho(h)$$

These are equal because characters are class functions, i.e.

$$\chi_i(hg^{-1}) = \chi_i(g^{-1}h)$$

This proves that P_i is a map of G-reps. □

Proof of (2). Suppose that $V = V_j$ an irreducible representation. Consider $P_i : V_j \to V_j$. What we want to show is that

$$\text{Image}(P_i) = \begin{cases} \langle 0 \rangle & \text{if } i \neq j \\ V_j & \text{if } i = j \end{cases}$$

By (1), we know that P_i is a map of irreducible G-representations, so by Schur's Lemma, $P_i = c \cdot \text{Id}_{V_j}$ for some c; i.e. P_i is some scaling of the identity. Now, we claim that

$$\text{Tr}(P_i) = |G| \cdot \langle \chi_i, \chi_j \rangle$$

By definition, we have

$$\begin{aligned} \text{Tr}_{V_j}(P_i) &= \text{Tr}_{V_j}\left(\sum_{g \in G} \overline{\chi_i(g)}\rho(g)\right) \\ &= \sum_{g \in G} \overline{\chi_i(g)}\, \text{Tr}_{V_j}(\rho(g)) \\ &= \sum_{g \in G} \overline{\chi_i(g)}\chi_j(g) \\ &= |G|\langle \chi_i, \chi_j \rangle. \end{aligned}$$

As a consequence, we must have that

$$(\dim V_j) \cdot c = |G| \cdot \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Therefore if $i \neq j$, then the image is $\langle 0 \rangle$ (because the constant is 0), and if $i = j$ then the image is all of V_j (because the map is a nonzero multiple of the identity).

Now, we claim that P_i is "functorial" in the sense that the following diagram commutes for any map of G-representations φ:

$$\begin{tikzcd}
(W,\sigma) \arrow[r, "\varphi"] \arrow[d, "P_i"] & (V,\rho) \arrow[d, "P_i"] \\
(W,\sigma) \arrow[r, "\varphi"] & (V,\rho)
\end{tikzcd}$$

Formally, P_i is an endomorphism of the identity functor of $\text{Rep}(G)$. We prove this by unraveling the definitions, as per usual:

$$\begin{aligned}
P_i \circ \varphi &= \left(\sum_{g \in G} \overline{\chi_i(g)} \sigma(g) \right) \circ \varphi \\
&= \sum_{g \in G} \left(\overline{\chi_i(g)} \sigma(g) \circ \varphi \right) \\
&= \sum_{g \in G} \left(\varphi \circ \left(\overline{\chi_i(g)} \rho(g) \right) \right) \\
&= \varphi \circ \left(\sum_{g \in G} \overline{\chi_i(g)} \rho(g) \right) \\
&= \varphi \circ P_i
\end{aligned}$$

Where we moved from the second to the third line by using the assumption that φ is a map of G-representations Let us come back to (V, ρ). Suppose that $V \cong \bigoplus_j V_j^{\oplus d_j}$. Denote by $\varphi_j^\ell : V_j^\ell \hookrightarrow V$ the inclusion maps for $\ell = 1, \dots d_j$. Now, applying the naturality to φ_j^ℓ, we have that

$$\begin{tikzcd}
V_j^\ell \arrow[r, "\varphi_j^\ell"] \arrow[d, "P_i"] & V \arrow[d, "P_i"] \\
V_j^\ell \arrow[r, "\varphi_j^\ell"] & V
\end{tikzcd}$$

commutes. It follows that the P_i on the right is an isomorphism on the V_i components of V (that is, when the inclusion map is φ_i^ℓ for some ℓ) and vanishes elsewhere. This concludes the proof. $\square$

Proof of (3). Since

$$P_i = c \cdot \text{Id}_{V_i} = \frac{|G|}{\dim V_i} \text{Id}_i$$

(3) follows immediately. $\square$

This marks the end of representation theory, and the beginning of rings and modules.

Definition 23.2. A **ring** R is a set with two operations $+, \cdot : R \times R \to R$ (the former which we call addition and the latter multiplication), such that

1. R with addition is an Abelian group, whose additive identity we denote by 0,
2. R with multiplication is a monoid with an identity which we denote by 1 (that is, it is associative but not necessarily commutative, and inverses need not exist), and
3. multiplication distributes over addition.

When multiplication is commutative, we refer to the ring as a **commutative ring**.

Example 23.3.
1. Any field is a ring.
2. $R = \mathbb{Z}$ the set of integers is a very important ring (one may recall from the midterm that it is the initial object in the category of rings).
3. $\mathbb{C}[x_1, \dots, x_n]$
4. $R = M_n(\mathbb{C})$. That is, the ring of complex matrices where $+$ is entrywise addition and $\cdot$ is matrix multiplication.

Definition 23.4.
1. A **ring map** $c : K \to R$ is a set map satisfying the following axioms.

$$
\begin{aligned}
c(k_1 + k_2) &= c(k_1) + c(k_2) \\
c(k_1 \cdot k_2) &= c(k_1) \cdot c(k_2) \\
c(0) &= 0 \\
c(1) &= 1.
\end{aligned}
$$

2. If K is a field, then a K-algebra A is a ring with a ring map $c : K \to A$.

Exercise 23.5.
1. If A is not the zero ring, the map c is an injection.
2. An equivalent definition is that A is a K-vector space with respect to addition and multiplication is K-linear.

Example 23.6. Some common $\mathbb{C}$-algebras include $\mathbb{C}, \mathbb{C}[x_1, \dots, x_n]$, and $M_n(\mathbb{C})$.

Example 23.7 (Key Example)**.** Let G be a finite group. Let $A = \mathbb{C}[G] = \{f : G \to \mathbb{C}\}$ be a $\mathbb{C}$-algebra, which we commonly refer to as the group algebra. The operations of this algebra is usual addition for $+$ and convolution for $\cdot$. The convolution is defined in the following way:

$$(f_1 * f_2)(x) = \left(\left(\sum_{g \in G} f_1(g)\delta_g\right) * \left(\sum_{g \in G} f_2(g)\delta_g\right)\right)(x) = \sum_{g_1 g_2 = x} f_1(g_1)f_2(g_2).$$

In other words, we have $f_1 * f_2 = \sum_{x \in G} \left(\sum_{g_1 g_2 = x} f(g_1)f_2(g_2)\right)\delta_x$.

In the special case that $f_1 = \delta_{g_1}$ and $f_2 = \delta_{g_2}$, we have

$$f_1 * f_2 = \delta_{x_1} * \delta_{x_2} = \sum_{x \in G} \left(\begin{cases} 1 & \text{if } g_1 = x_1, g_2 = x_2 \\ 0 & \text{otherwise} \end{cases}\right)\delta_x = \delta_{x_1 x_2}.$$

Therefore, we can think of convolution as $\mathbb{C}$-linear extension of group multiplication. Note that this is different from the pointwise multiplication of functions.

Exercise 23.8.

1. $\mathbb{C}[G]$ is commutative if and only if G is Abelian.
2. The ring map $c : \mathbb{C} \to C[G]$ is defined by $c(z) = z\delta_e$.

The next idea we wish to introduce is the idea of a module.

Definition 23.9. Given a ring R, an R-module M is a set with structures

1. M with addition is an Abelian group,
2. the action map $R \times M \to M$ is associative and unital, and
3. the action distributes over addition.

Alternatively, we can reformulate the idea of an R-module as a ring map $R \to \mathrm{End}_{\mathrm{Ab}}(M)$.

Example 23.10.

1. If R is a field K, then an R-module M is a K-vector space.
2. If R is $\mathbb{Z}$, then R-modules are Abelian groups.
3. If R is the polynomial algebra, then R-modules are $\mathbb{C}$-vector spaces M equipped with n commuting endomorphisms $x_i : M \to M$ for $i = 1, \ldots, n$. Algebraic geometry is essentially the study of these $\mathbb{C}[x_1, \ldots, x_n]$-modules.

Theorem 23.11. *If $R = M_n(\mathbb{C})$, then the R-modules are all direct sums of copies of the vector module $\mathbb{C}^n$.*

Exercise 23.12. If $R = \mathbb{C}[G]$ is the group algebra, R-modules are complex G-representations.

We shall show this exercise next week.