Today: Other central characters. Previously, we have been working with
the trivial central character \( \chi_0 \) (i.e. the char of the trivial rep).
We will continue with \( \mathfrak{sl}_2 \).

Recall: \( \mathfrak{usl}(2)/U \cong \mathbb{C}[c] \)

Concretely: \( \mathfrak{usl}(2)/U \cong \mathbb{C} U/\mathbb{C} U = U/U = \mathbb{C}[c] \)

Universal principal series

Identifies \( \mathfrak{g} \cong \mathbb{C}[[H]]^k \).

\[ \text{Spec } \mathbb{C}[H] \]
\[ \text{Spec } \mathbb{C}[[H]]^k \]

What does \( U_3 = \mathfrak{usl}(3) \otimes \mathbb{C} \mathfrak{g} \mathfrak{l} \) -modules? In particular \( h \)-w modules

Answer: We will draw \( h^* \) which lets us draw the "lift" of \( x \)

**Case 1:** \( x \in \mathbb{C}[[H]] \)

\[ (U_{x}, N) \text{-mod } \]

where \( x \) is image of \( \lambda \).

\[ \lambda = (U_0, N) \text{-mod} \]

\[ L_0 \leftrightarrow L_0 \]
\[ V_0 \leftrightarrow V_0 \]
\[ V_{n-2} \leftrightarrow V_{n-2} \]
Case 2: $\lambda \neq \mathbb{N}$ (generic)

\[ (X, \mathfrak{m}) - \text{mod} \cong \text{Vect} \oplus \text{Vect} \oplus \text{semisimple} \]

\[ V_{\lambda} \oplus V_{\lambda-2} \]

simple & induced

Case 3: $\lambda = -1$

\[ (X, \mathfrak{m}) - \text{mod} \cong \mathcal{E}[\mathbb{E}] - \text{mod} \text{ with } \mathbb{E} = 0 \]

\[ V_1 \text{ simple & induced} \]

has one non-trivial ext. by itself

**BB localization**

Idea: "geometrize" univ. principal series $\mathcal{U}(\mathfrak{sl}(2))/\mathcal{U}(\mathfrak{n})$ via $G_m$ action.

Consider $A^2 - \mathfrak{e}_0 = \mathfrak{m}$, $\mathcal{U}(\mathfrak{sl}(2))/\mathcal{U}(\mathfrak{n}) \cong B/N = 1$.

\[ \mathbb{P}^1 \]

We have $\mathcal{U}(\mathfrak{sl}(2)) \to \mathbb{C} \left( A^2 - \mathfrak{e}_0, \mathfrak{m} = \mathfrak{e}_0 \right)$

Can calculate $D_{\mathbb{P}^1}$ via Har. reduction from $D(A^2 - \mathfrak{e}_0)$

Classically:

\[ T^* (A^2 - \mathfrak{e}_0) \xrightarrow{\mathbb{P}^1} T^* \mathbb{P}^1 \times (A^2 - \mathfrak{e}_0) \xrightarrow{T^* \pi} T^* \mathbb{P}^1 \]

\[ \mathbb{P}^1 \]

\[ \mathbb{P}^1 \]

Left invariant fibr. of $T^*$
quantize: \[ D_{\pi} = \left( D_{A^2,E^3} / \langle \text{vector fields along scaling} \ H\text{-action} \rangle \right)^{H\text{-action invariance}} \]

Now for \( A \in H^* \), can introduce \( A \)-twisted diff. operators \( D_{\pi}^A \)

where

\[ D_{\pi}^A = \left( D_{A^2,E^3} / \langle H\text{-action} \rangle \right) \]

locally can find \( D_{\pi}^A \sim D_{\pi} \) (Not true that they are globally isom.)

BB localization: 1. Suppose \( \lambda \in \mathbb{C}, z, 2, 3, \ldots \)

Then,

\[ u_{\lambda} \text{SL}(2) \sim \mod D_{\pi}^A - \mod \]

where \( \chi \) is rep.

2. \( \lambda \in \mathbb{C}, z, 2, 3, \ldots \)

Then,

\[ D \left( u_{\lambda} \text{SL}(2) \sim \mod \right) \sim D \left( D_{\pi}^A - \mod \right) \]

(Colored categories)

3. \( A = -1 \)

Then,

\[ u_{\lambda} \text{SL}(2) \sim \mod D_{\pi}^A - \mod / \langle O_{\pi}((-1)) \rangle \]

II. Riemann–Hilbert for twisted diff. maps \( (D_{\pi}^A, N)\)-mod

\[ D_{\pi}^A = \left( D_{A^2,E^3} / \langle H\text{-action} \rangle \right) \]

Regard it as a sheaf of \( \mathfrak{g} \)-algebras on \( A^2 \times \mathbb{R}^3 \).

Fix \( \lambda \) set \( m = e^{2\pi i \lambda} \)

Consider const. sheaves \( \mathbb{C}[x]/(x^{m+1}) \) on \( A^2 \times \mathbb{R}^3 \).

the coh. sheaves restrict to as line \( \lambda \) origin

are local systems

with monodromy \( \lambda \) m.
LH

$\left( D_{\mathfrak{p}^1}, N \right)$-mod $\to D_c^1 (A^2 \times \mathbb{S}^3)$

Let's analyze the $D_c^1 (A^2 \times \mathbb{S}^3)$ for generic $a$ (i.e. not in $\mathbb{Z}$).

Cohom sheaves will be local systems on two pieces (which are the $B$-orbits).

Claim: Two simple objects and no interaction

1) $L_0 [1]$ local system on $A_0$ with monodromy $m = e^{2\pi i \alpha}$

2) $L_1 [2]$ local system on $A_1$ with monodromy $m = e^{2\pi i \beta}$

Exercise: No maps or extensions between the two.

This shows that the cat is vector.
Equivalence \((D_{\pi}(N) - \text{mod} \equiv (D_{\pi}(N) - \text{mod} \text{ when } t-n \in \mathbb{Z})\)

Note: \(D_{\pi}(c^{2\pi}) \text{ only depends on } m = e^{2\pi i \lambda}\)