Langlands:

Today: Other central characters. Previously, we have been working with the trivial central character χ_0 (i.e. the char of the trivial rep)

We will continue with $sl_2 \mathbb{C}$.

Recall: $Usl(2) \supset \mathfrak{z} \simeq \mathbb{C}[c]$
center ↑ casimir ↑

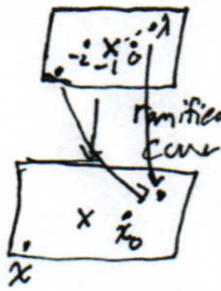
Concretely: $Usl(2) \circlearrowright Usl(2)/U\mathfrak{n} \circlearrowleft U\mathfrak{b}/U\mathfrak{n} \simeq U\mathfrak{h} = \mathbb{C}[H]$

Universal principal series

Recall: $\mathfrak{n}$ = generated by X
$\mathfrak{b}$ = generated by H and X
$\mathfrak{h}$ = gen by H

Identifies $\mathfrak{z} \simeq \mathbb{C}[(H+1)^2]$.

$$\rightsquigarrow \quad \begin{array}{c} \operatorname{Spec} \mathbb{C}[H] \\ \downarrow \\ \operatorname{Spec} \mathbb{C}[(H+1)^2] \end{array}$$



$\mathfrak{h}^*$

What are $U_\chi = Usl(2) \otimes_{\mathfrak{z}} \mathbb{C}_\chi$ -modules? In particular h-w modules

Answer: We will draw $\mathfrak{h}^*$ which lets us draw the "lift" of χ

Case 1: $\lambda \in \mathbb{N}$

(U_χ, N)-mod (as before)

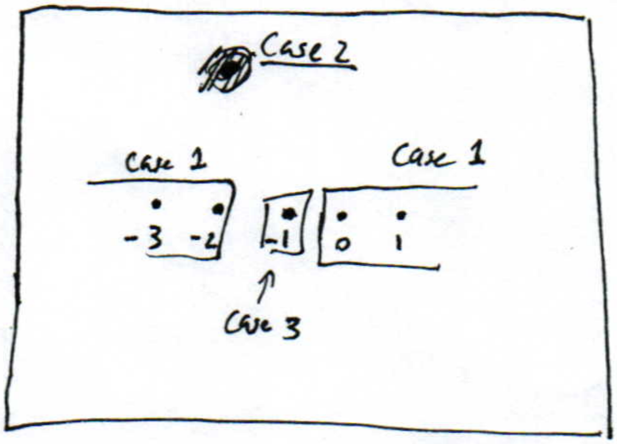
when χ is image of λ.

$\simeq (U_0, N)$-mod

$L_n \longleftrightarrow L_0$

$V_n \longleftrightarrow V_0$

$V_{n-2} \longleftrightarrow V_{-2}$



$\mathfrak{h}^*$

Case 2: $\lambda \notin \mathbb{Z}$ (generic)

$$(\mathcal{U}_{\chi}, N)\text{-mod} \simeq \underset{V_\lambda}{\mathrm{Vect}} \oplus \underset{V_{\lambda-2}}{\mathrm{Vect}} \simeq \mathbb{C}[\varepsilon]\text{-mod with } \varepsilon^2 = 1$$

e semisimple

simples & induced

Case 3: $\lambda = -1$ $(\mathcal{U}_{\chi}, N)\text{-mod} \simeq \mathbb{C}[\varepsilon]\text{-mod with } \varepsilon^2 = 0$

V_{-1} simple & induced

has one non-trivial ext. by itself

BB localization

Idea: "geometrize" univ. principal series $\mathcal{U}sl(2)/\mathcal{U}n \circlearrowleft \mathcal{U}h$

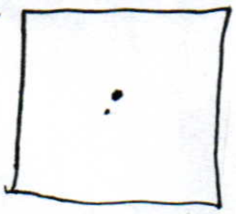
$\curvearrowright \mathcal{U}sl(2)$ left-action

scaling action by $\mathbb{G}_m$.

Consider $SL(2) \circlearrowright \mathbb{A}^2 - \{0\} \simeq SL(2)/N \circlearrowleft B/N \simeq H$

$$\downarrow \text{ quotient by scaling action.}$$

$\mathbb{P}^1$



$\mathbb{A}^2 - \{0\}$

We have: $\mathcal{U}sl(2) \longrightarrow \Gamma(\mathbb{A}^2 - \{0\}, \mathcal{D}_{\mathbb{A}^2 - \{0\}})$

Can calculate $\mathcal{D}_{\mathbb{P}^1}$ via Ham. reduction from $\mathcal{D}_{\mathbb{A}^2 - \{0\}}$

Classically: $T^*(\mathbb{A}^2 - \{0\}) \xleftarrow{q^*} T^*\mathbb{P}^1 \times_{\mathbb{P}^1} (\mathbb{A}^2 - \{0\}) \xrightarrow{\pi} T^*\mathbb{P}^1$

$$\underset{\text{quotient}}{\mathcal{O}_{T^*(\mathbb{A}^2 - \{0\})}} \xrightarrow{q^*} \mathcal{O}_{T^*\mathbb{P}^1 \times_{\mathbb{P}^1} (\mathbb{A}^2 - \{0\})} \underset{\text{sub invariant fibers of } \pi}{\xhookleftarrow{\pi^*}} \mathcal{O}_{\mathbb{P}^1}$$

<u>quantize:</u> $$D_{\mathbb{P}^1} = \left(D_{\mathbb{A}^2-\{0\}} \Big/ \langle \text{vector fields along scaling } H\text{-action} \rangle \right)^{H\text{-action invariance}}$$

(H) = ideal of 0 in $\mathbb{C}[\mathfrak{h}]$

Now <u>for $\lambda \in \mathfrak{h}^*$</u>, can introduce ~~a~~ <u>λ-twisted</u> diff. operators. $D_{\mathbb{P}^1}^{\lambda}$

where
$$D_{\mathbb{P}^1}^{\lambda} = \left(D_{\mathbb{A}^2-\{0\}} \Big/ (H-\lambda) \right)^{H\text{-action invariance}}$$

locally can find $D_{\mathbb{P}^1}^{\lambda} \simeq D_{\mathbb{P}^1}$ (Not true that they are globally iso)

<u>BB localization</u> 1. Suppose $\lambda \notin \{-1,-2,-3,\dots\}$

Then,
$$\mathcal{U}_{\chi}\mathfrak{sl}(2)\text{-mod} \underset{\text{global sec}}{\overset{\text{localization}}{\rightleftarrows}} D_{\mathbb{P}^1}^{\lambda}\text{-mod}$$
where χ is rep. by λ.

2. $\lambda \in \{-2,-3,\dots\}$ Then,
$$D\left(\mathcal{U}_{\chi}\mathfrak{sl}(2)\text{-mod}\right) \simeq D\left(D_{\mathbb{P}^1}^{\lambda}\text{-mod}\right)$$
(derived categories)

This is a gen. of Borel-Weil-Bott

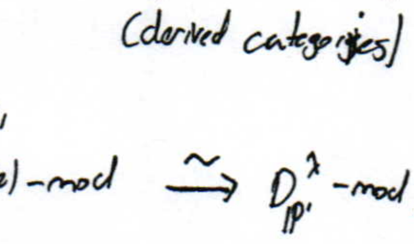
3. $\lambda = -1$ Then,
$$\mathcal{U}_{\chi}\mathfrak{sl}(2)\text{-mod} \xrightarrow{\sim} D_{\mathbb{P}^1}^{\lambda}\text{-mod} \big/ \langle \mathcal{O}_{\mathbb{P}^1}(-1) \rangle$$

T <u>Riemann-Hilbert for twisted diff mods</u> $(D_{\mathbb{P}^1}^{\lambda}, N)$-mod
$$D_{\mathbb{P}^1}^{\lambda} = \left(D_{\mathbb{A}^2-\{0\}} \big/ (H-\lambda) \right)^{H}$$
Regard it as a sheaf of ~~sheg~~ algebras over $\mathbb{A}^2-\{0\}$.

<u>Fix λ</u> <u>set $m = e^{2\pi i \lambda}$</u>

call this cat. $D_{\mathbb{C}}^{\lambda}(\mathbb{A}^2-\{0\})$

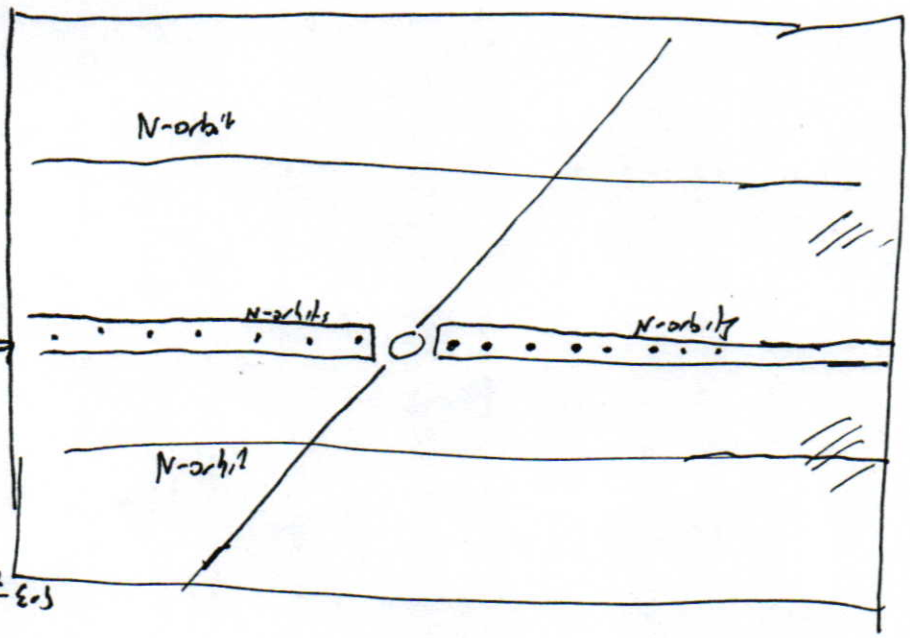
Consider const. complexes ~~sheaves~~ ~~[illegible]~~ ~~[illegible]~~ on $\mathbb{A}^2-\{0\}$ s.t.
the cohom sheaves restricted to any line $\ni$ origin are local systems with monodromy m.



<u>RH</u>

$$\begin{array}{ccc} (D^{\lambda}_{\mathbb{P}^1}, N)\text{-mods} & \hookrightarrow & D^{\lambda}_c(\mathbb{A}^2 \setminus \{0\}) \\ & \searrow^{\cong} & \cup \\ & & \mathrm{Perv}^{\lambda}_N(\mathbb{A}^2 \setminus \{0\}) \end{array}$$

Let's analyze the $\mathrm{Perv}^{\lambda}_N(\mathbb{A}^2 \setminus \{0\})$ for generic λ (i.e. not in $\mathbb{Z}$)



$\mathbb{A}^2 \setminus \{0\}$

$A_0 =$ local system on line $\cong \mathbb{G}_m$

with inclusion $A_0 \overset{i}{\hookrightarrow} A_1 \overset{j}{\hookrightarrow} \mathbb{A}^2 - \{0\}$

local system on comp of line $= A_1 \cong \mathbb{A}^1 \times \mathbb{G}_m$

Cohom sheaves will be local systems on two pieces (which are the B-orbits)

= inverse image of schubert cells

<u>Claim</u>: Two simple objects and no interaction

1.) $\mathcal{L}_0[1]$ local system on A_0 with monodromy $m = e^{2\pi i \lambda}$

2.) $\mathcal{L}_1[2]$ local system on A_1 with monodromy $m = e^{2\pi i \lambda}$

<u>Exercise:</u> No maps or extensions between the two,

This shows that the cat is $\mathrm{Vect} \oplus \mathrm{Vect}$.

Equivalence $(D^{\lambda}_{\mathbb{P}^1}, N)$-mod $\simeq (D^{\mu}_{\mathbb{P}^1}, N)$-mod when $\lambda - \mu \in \mathbb{Z}$

Note: $D^{\lambda}_{c}(\mathbb{A}^2 - \{0\})$ only depends on $m = e^{2\pi i \lambda}$