2/14/17 **Perverse Sheaves on Curves** "Neither perverse nor a sheaf"

X smooth curve $\rightsquigarrow$ $\mathrm{Perv}(X)$ abelian category $\subset D(X)$ derived cat. of shvs on X

In fact, $\mathrm{Perv}(X) \underset{\text{full}}{\subset} D_c(X)$ = derived cat. of constructible sheaves on X — not abelian, but triangulated

Def

A complex $\mathcal{F}$ on X is constructible if there is open $U \subset X$, closed $Y = X \setminus U$ s.t. $H^\bullet \mathcal{F}|_U$ is locally constant w/ FD stalks (finite rank local system)

$H^\bullet$ cohomology functor (commutes w/ restriction to U)

and $H^\bullet \mathcal{F}|_Y$ is FD.

i.e can cut up X into pieces and $H^\bullet \mathcal{F}$ doesn't jump between pieces

Ex

$X = \mathbb{P}^1$

1) constant sheaf $\mathbb{C}_{\mathbb{P}^1}[n]$ shifted by $n \in \mathbb{Z}$.

2) Hopf fibration: $S^3 \xrightarrow{\pi} \mathbb{P}^1$ (or any circle bundle)

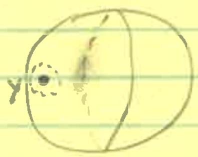
$R\pi_* \mathbb{C}_{S^3}[n]$ push forward in derived cat.

3) $Y = \{0\} \overset{i}{\hookrightarrow} \mathbb{P}^1$, $U = \mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1$. $i_* \mathbb{C}_{\{0\}}[n]$. (dropping R but implicitly there)

4) $j_* \mathbb{C}_{\mathbb{A}^1}[n]$. can compute

$i^* j_* \mathbb{C}_{\mathbb{A}^1}[n] = C^\bullet\left(\underbrace{\text{punctured disk}}_{\text{intersection of open disk around } Y \text{ w/ } \mathbb{A}^1}\right)[n] \rightsquigarrow H^\bullet = H^\bullet(S^1)[n]$

cochains



5) $j_! \mathbb{C}_{\mathbb{A}^1}[n]$. $i^* j_! \mathbb{C}_{\mathbb{A}^1}[n] = C^\bullet(\odot, \circ)[n] \rightsquigarrow H^\bullet = 0$

cochains on annulus relative to Y

$Y = \{0, \infty\} \overset{i}{\hookrightarrow} \mathbb{P}^1$, $U = \mathbb{G}_m \overset{j}{\hookrightarrow} \mathbb{P}^1$

6) j_* or $j_!$ of $\mathcal{L}_{\mathbb{G}_m}[n]$ ← fin. rank local system

Def

$\mathrm{Perv}(X) \subset D_c(X)$ consists of $\mathcal{F}$ s.t., for which $U \overset{j}{\hookrightarrow} X$ open, $Y = X \setminus U \overset{i}{\hookrightarrow} X$ closed on which $\mathcal{F}$ is constructible, we have $H^\bullet \mathcal{F}|_U$ is concentrated in deg -1

and $H^\bullet(i^* \mathcal{F})$ conc. in deg $-1, 0$ "$H^\bullet(j^* \mathcal{F})$"

$H^\bullet(i^! \mathcal{F})$ conc. in deg $0, 1$ "$H^\bullet(j^! \mathcal{F})$"

Next

Thm that $\mathrm{Perv}(X)$ is actually abelian.

Rmk

$i^* =$ "sections near Y"

$i^! =$ "sections supp on Y (zero off Y)"

5 fav. ex

$X = \mathbb{P}^1$, $Y = \{0\}$, $U = \mathbb{A}^1$

1) $\mathbb{C}_{\mathbb{P}}[1]$ $\leftarrow$ $H^\bullet(i^! \mathcal{F})$ has a deg 2-cochain, but we shifted down to deg 1, so $H^\bullet(i^!\mathcal{F})$ conc. in deg 2

$i^* \mathbb{C}_{\mathbb{P}}[1]$ = restriction to Y = $\mathbb{C}_{\{0\}}[1]$, so conc. in deg -1

2) $\mathbb{C}_{\{0\}}$ has $i^* \mathbb{C}_{\{0\}} = i^! \mathbb{C}_{\{0\}} = \mathbb{C}_{\{0\}}$, so all conc in deg 0.

3) $j_* \mathbb{C}_{\mathbb{A}^1}[1]$ $\quad i^* j_* \mathbb{C}_{\mathbb{A}^1}[n] \simeq C^\bullet(\text{annulus})[1]$ which is concentrated in deg $-1, 0$.

$i^! j_* \mathbb{C}_{\mathbb{A}^1}[n] \simeq C^\bullet(\text{annulus, outside boundary})[1] \simeq 0$
$\uparrow$ quasi-ism.

4) $j_! \mathbb{C}_{\mathbb{A}^1}[1]$. $\quad i^* j_! \mathbb{C}_{\mathbb{A}^1}[1] \simeq C^\bullet(\text{annulus, inside bndry})[1] \simeq 0$

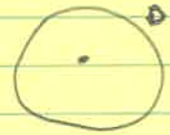
$i^! j_! \mathbb{C}_{\mathbb{A}^1}[1] \simeq C^\bullet(\text{annulus, inside bnd} \sqcup \text{outside bnd})[1]$

$\simeq C^\bullet(\text{annulus})$ b/c $\{C^\bullet(\text{annulus, bnd}) \gg$ shifted up by 1

$[1]$ shifts back down

5) Exercise: Check T is perverse.
$\uparrow$ indecomposable, under Riemann-Hilbert.

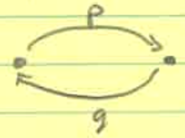
Back to quiver description, now for $\{0\} \subset D$ disk



Perv(D) constr wrt $\{0\}$, $D^\times = D \setminus \{0\}$

Perv$(D, 0)$:=

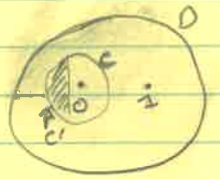
Thm $\quad$ Perv$(D, \{0\}) \simeq$ F.D. modules over



s.t. $1 - pq$, $1 - qp$ invt.

Note

From this, we can see why Perv$(D, 0)$ is abelian cat, since it is just modules.



Construction of functor Perv$(D, \{0\}) \longrightarrow$ FD mod. $\qquad \{1\} \xrightarrow{e} D$

$\mathcal{F} \longmapsto$

coboundary map of LES $\quad \xrightarrow{\delta}$

$\Gamma(C', \mathcal{F})[-1] \simeq e^* \mathcal{F}[-1] \quad \Gamma(C, C', \mathcal{F})$ = sections of $\mathcal{F}$ on C vanishing on C'

$\uparrow \simeq$ $\qquad$ restriction to

$\Gamma(B, B \cap C', \mathcal{F})$

(B relative to $B \cap C'$)

$\simeq$ $\qquad$ III