Topics in algebra
Valentine's day edition

We will define and study perverse sheaves on curves. Consider some smooth curve X. We will attach to X an abelian category $Perv(X)$ (... remember - it is Valentine's day...), which is contained in derived category of *constructible sheaves* on X. We begin with explaining what adjective *constructible* means.

Definition 1. A complex $\mathcal{F}$ on X if there exists an open $U \subseteq X$ and closed $Y = X \setminus U$ such that

- $H^\bullet \mathcal{F}|_U$ is locally constant with finite dimensional stalks (i.e. finite rank local system),
- $H^\bullet \mathcal{F}|_Y$ is finite dimensional (Y is contains only finitely many points, so "locally" does not correspond).

Example 2. Most of the time we focus on the $\mathbb{P}^1$, therefore we give examples of constructible sheaves for $X = \mathbb{P}^1$.

(1) $\mathbb{C}_{\mathbb{P}^1}[n]$, where $n \in \mathbb{Z}$ (note that for $n \neq 0$ sheave $\mathbb{C}_{\mathbb{P}^1}[n]$ lives in derived category $D(\mathbb{P}^1)$),
(2) let M^3 be a 3-manifold and let $\pi : M^3 \to \mathbb{P}^1$ be onto (so called *circle bundle*, fiber bundle with fiber equal to S^1, eg. Hopf fibration), then $\mathcal{R}\pi_*\mathbb{C}_{M^3}[n]$, where $n \in \mathbb{Z}$, is constructible,
(3) for $i : Y := \{0\} \to \mathbb{P}^1$ and $j : U := \mathbb{A}^1 \to \mathbb{P}^1$ it follows that $i_*\mathbb{C}_Y[n]$, where $n \in \mathbb{Z}$ is constructible,
(4) i, j, U an Y as above, then $j_*\mathbb{C}_U[n]$, where $n \in \mathbb{Z}$, is constructible and we have
$$i^*j_*\mathbb{C}_U[n] = C^*((\circ))[n], \qquad H^\bullet((\circ)) \cong H^\bullet(S^1)[n],$$
(5) i, j, U an Y as above $j_!\mathbb{C}_{U1}[n]$, where $n \in \mathbb{Z}$, is constructible,
$$i^*j_!\mathbb{C}_U[n] = C^\bullet(\circledcirc, \circ)[n], \qquad H^\bullet(\circledcirc, \circ) \cong 0,$$
(6) $i : Y := \{0, \infty\} \to \mathbb{P}^1$, $j : U := \mathbb{G}_m \to \mathbb{P}^1$, then we can consider j_* or $j_!$ of $\alpha_{\mathbb{G}_m}[n]$ (finite rank local system), where $n \in \mathbb{Z}$.

Definition 3. A full subcategory of $D_C(X)$ (constructible guys in $D(X)$), denoted by $Perv(X)$, consists of $\mathcal{F}$ such that there exist an open $U \subseteq X$, a closed $Y = X \setminus U$ and embeddings $j : U \to X$ and $i : Y \to X$ for which $\mathcal{F}$ is constructible, moreover

- $H^\bullet(j^*\mathcal{F}) = H^\bullet(j^!\mathcal{F})$ is concentrated in degree -1,
- $H^\bullet(i^*\mathcal{F})$ and $H^\bullet(i^!\mathcal{F})$ are concentrated in degree 0 and degree 1.

Remark 4. (1) i^* means "sections near Y",
(2) $i^!$ means "sections supported on Y",
(3) $Perv(X)$ is abelian (not so easy to show).

Example 5 ("almost" 5 favourite examples)**.** Assume that $X = \mathbb{P}^1$, $Y = \{0\}$ and $U = \mathbb{A}^1$.

(1) For $\mathbb{C}_{\mathbb{P}^1}[1]$ we have

$$i^*\mathbb{C}_{\mathbb{P}^1}[1] \cong \mathbb{C}_Y[1], \qquad i^!\mathbb{C}_{\mathbb{P}^1}[1] \cong \mathbb{C}_Y[1].$$

(2) For $\mathbb{C}_Y$ we have

$$i^*\mathbb{C}_Y = i^!\mathbb{C}_Y = \mathbb{C}_Y.$$

(3) For $j_*\mathbb{C}_U[1]$ we have

$$i^*j_*\mathbb{C}_U[1] = C^*(\circ)[1],$$

$$i^!j_*\mathbb{C}_U[1] = C^\bullet(\circledcirc, \bigcirc)[1] \cong 0.$$

(4) For $j_!\mathbb{C}_U[1]$ we have

$$i^*j_!\mathbb{C}_U[1] \cong C^\bullet(\circledcirc, \circ)[1] \cong 0,$$

$$i^!j_!\mathbb{C}_U[1] \cong C^*(\circledcirc, \circ \sqcup \bigcirc)[1] \cong C^*(\circledcirc),$$

concentrated in degrees 0 and 1.

Exercise: Check that T is perverse.

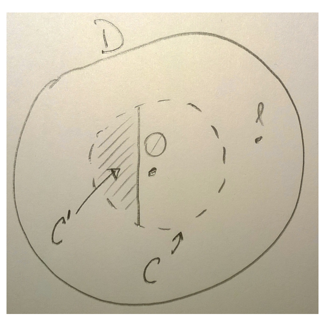
Now to the description of quiver: $\{0\} \subseteq D$, where D is a disk. We see that $Perv(D, \{0\})$ is constructible for $Y = \{0\}$ and $U = D^*$.

Theorem 6. *$Perv(D\{0\}) \cong$ finite dimensional modules over*

$$\bullet \underset{q}{\overset{p}{\rightleftarrows}} \bullet \ ,$$

such that $1 - pq$ and $1 - qp$ are invertible.

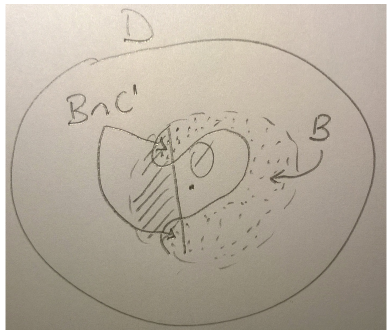
Sketch of the proof - construction of the functor. We start with a disk D, denote 0, 1 and sets C and C',



Let e be the inclusion $\{1\} \to D$, $\Gamma(C, C', \mathcal{F})$ = "sections of $\mathcal{F}$ on C vanishing on C'", $\Gamma(C', \mathcal{F})[-1] \cong e^*\mathcal{F}[-1]$. In this situation

$$\Gamma(C', \mathcal{F})[-1] \xrightarrow{\ p\ } \Gamma(C, C', \mathcal{F}) \ ,$$

where p is co-boundary of LES. Consider a set $B \subseteq C$



and a map given by the restriction $\Gamma(C, C', \mathcal{F}) \to \Gamma(B, B \cap C', \mathcal{F}) \cong \Gamma(C', \mathcal{F})[-1]$, say q. We see have

$$\Gamma(C', \mathcal{F})[-1] \underset{q}{\overset{p}{\rightleftarrows}} \Gamma(C, C', \mathcal{F}) \ ,$$

and maps $1 - qp$ and $1 - pq$ are monodromies.

†Instytut Matematyczny, Uniwersytet Wrocławski, Wrocław, Poland
E-mail address: `daniel.hoffmann@math.uni.wroc.pl`