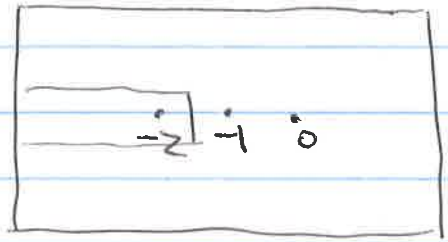
2/9/2017

Recall: Bases for $(U_0\mathfrak{sl}(2), N)$-mods.

1) Simples : L_0, V_{-2}



Spec $\mathbb{C}[H]$

2) Vermas : V_0, V_{-2}
(costandards)

3) coVermas : $V_0', V_{-2}' = V_{-2}$.
(standards)

Remark: relating 1) $\leftrightarrow$ 2), 3) is Kazhdan-Lusztig theory.

4) Projectives : $V_0 \to L_0$ $\quad T = P_{\mathbb{A}^1}/D_{\mathbb{A}^1}(z\partial_z z) \to V_{-2}$

indecomposables : $L_0, V_{-2}, V_0, V_0', T$.

5) Injectives : $L_0 \to V_0'$ $\quad V_{-2} \to T$

Use Projectives to ~~generate~~ give quiver presentation :

Proj. generator : $V_0 \oplus T =: P$.

$(U_0(\mathfrak{sl}(2)), N)$-mods $\longrightarrow$ End(P)-mods

$M \longrightarrow \mathrm{Hom}(P, M)$

$$\begin{tikzcd} \mathrm{Hom}(V_0,M) \arrow[r, bend right, "q^*"'] & \mathrm{Hom}(T,M) \arrow[l, bend right, "p^*"'] \end{tikzcd}$$

(V_0) $\quad$ (T)

$\mathrm{End}(V_0) = \mathbb{C}$, $q^* p^* = 0$. $\quad$ $\mathrm{End}(T) = \mathbb{C}[\varepsilon]/(\varepsilon^2)$

$\varepsilon = p^* q^*$

$(\varepsilon^2 = 0)$

. there're natural morphisms.

$$p: \begin{tikzcd} T \arrow[r] \arrow[d, equal] & V_{-2} \arrow[r] \arrow[d, equal] & V_0 \arrow[d, equal] \\ D_{\mathbb{A}^1}/(z\partial_z z) & D_{\mathbb{A}^1}/(z) & D_{\mathbb{A}^1}/(z\partial_z) \end{tikzcd}$$

$$\begin{tikzcd} D_{\mathbb{A}^1}/(z\partial_z z) \arrow[r] & D_{\mathbb{A}^1}/(z) \arrow[r] & D_{\mathbb{A}^1}/(z\partial_z) \\ & 1 \arrow[r, mapsto] & \partial_z \end{tikzcd}$$

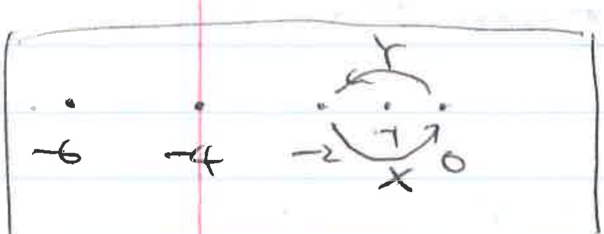
$$q: \begin{tikzcd} V_0 \arrow[r] \arrow[d, equal] & T \arrow[d, equal] \\ D_{\mathbb{A}^1}/(z\partial_z) & D_{\mathbb{A}^1}/(z\partial_z z) \\ 1 \arrow[r] & z \end{tikzcd}$$

Ex: Jordan-Holder series of T:

$0 \subset V_{-2} \subset V_0 \subset T$

$\uparrow V_{-2} \quad \uparrow L_0 \quad \uparrow V_{-2}$

- Find quiver back in reps



- $XY = 0$
- $(YX)^2 = 0$

$\rightsquigarrow$ Indecomposables :

1) L_0 : $\overset{0}{\bullet}$ $\overset{\mathbb{C}}{\bullet}$

2) V_{-2} : $\overset{\mathbb{C}}{\bullet}$ $\overset{0}{\bullet}$

3) V_0 :
$$\begin{tikzcd} \overset{\mathbb{C}}{\bullet} \arrow[r, bend right, "0"'] & \overset{\mathbb{C}}{\bullet} \arrow[l, bend right, "\sim"'] \end{tikzcd}$$

4) V_0' :
$$\begin{tikzcd} \mathbb{C}\bullet \arrow[r, bend right, "\sim"'] & \bullet\mathbb{C} \arrow[l, bend right, "0"'] \end{tikzcd}$$

5). T :
$$\begin{tikzcd} \overset{\bullet}{\mathbb{C}\oplus\mathbb{C}} \arrow[r, bend left, "p_1"] & \overset{\bullet}{\mathbb{C}} \arrow[l, bend left, "i_2"] \end{tikzcd}$$

Find quiver geometrically via. Riem-Hilbert corresp.

Idea : D-modules $\rightsquigarrow$ Solutions
(spaces of sheaves of vector spaces)
--- Derived Category

Concretely on a curve X :

$$\underset{\text{(D-mods)}}{M} \longrightarrow \underset{\text{analytic Solutions}}{\left.\begin{matrix} \Omega^1_X\otimes M \\ \uparrow\nabla \\ M \end{matrix}\right]\begin{matrix} 0 \\ \\ -1 \end{matrix}} \leftarrow \text{de Rham cpx of } M.$$

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<u>We'll obtain:</u>

$$\begin{tikzcd} (\mathcal{D}_{\mathbb{P}^1}, N)\text{-mods} \arrow[r, leftrightarrow] \arrow[d, "\wr"] & D_N(\mathbb{P}^1) \\ \mathrm{Perv}_N(\mathbb{P}^1) \arrow[ur, hook] & \end{tikzcd}$$

$D_N(\mathbb{P}^1)$: derived cat. of N-equiv. sheaves on $\mathbb{P}^1$.

<u>Calculate for indecomposables:</u>

1) $L_0 \xrightarrow{BB} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{RH} \begin{array}{cc} & \text{deg} \\ \Omega^1_{\mathbb{P}} & 0 \\ \uparrow d & \\ \mathcal{O}^1_{\mathbb{P}} & -1 \end{array} \simeq \mathbb{C}_{\mathbb{P}^1}[1]$

2) $V_{-2} \xrightarrow{BB} \mathcal{D}_{\mathbb{A}^1}/(z) \xrightarrow{RH} \begin{array}{c} \mathbb{C}[\partial_z] \\ \uparrow \partial_z \\ \mathbb{C}[\partial_z] \end{array} \simeq \begin{array}{cc} & \text{deg} \\ \mathbb{C}_{\{0\}} & 0 \\ 0 & -1 \end{array}$

$\Gamma\Big[\ \mathcal{O}_{\mathbb{A}^1\setminus\{0\}}/\mathcal{O}_{\mathbb{A}^1} \xrightarrow{\partial} \mathcal{O}_{\mathbb{A}^1\setminus\{0\}}/\mathcal{O}_{\mathbb{A}^1} \otimes \Omega^1_{\mathbb{A}^1}$

$\qquad\qquad \wr \ dz$

$\qquad \xrightarrow{\partial_z} \mathcal{O}_{\mathbb{A}^1\setminus\{0\}}/\mathcal{O}_{\mathbb{A}^1}\ \Big]$

$\cdots \frac{1}{z^3} \xleftarrow{\partial_z} \frac{1}{z^2} \xleftarrow{\partial_z} \frac{1}{z}$

<u>Ex:</u>

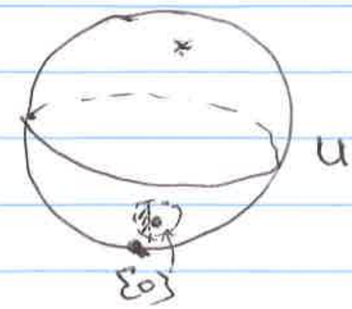
- $V_0 \xrightarrow{BB} \mathcal{D}_{\mathbb{A}^1}/(z\partial_z) \xrightarrow{RH} j_!\mathbb{C}_U[1]$

- $V_0' \xrightarrow{BB} \mathcal{D}_{\mathbb{A}^1}/(\partial_z z) \xrightarrow{RH} j_*\mathbb{C}_U[1]$ N-orbits: $U \overset{j}{\hookrightarrow} \mathbb{P}^1 \overset{i}{\hookleftarrow} \{0\}$

- $T \xrightarrow{BB} \mathcal{D}_{\mathbb{A}^1}/(z\partial_z z) \xrightarrow{RH}$ successive extension

Finally quiver presentation geometrically

$\mathbb{P}^1$:



stalk of cpx. (shifted up by 1)

boundary map in LES

sections over $\odot$ ( ) that vanish on ∂

relative sections of cpx

$$\times \underset{\text{restriction}}{\overset{\text{boundary map in LES}}{\rightleftarrows}} \bullet$$

~~res~~ restriction