Langlands duality
Lec. 07.

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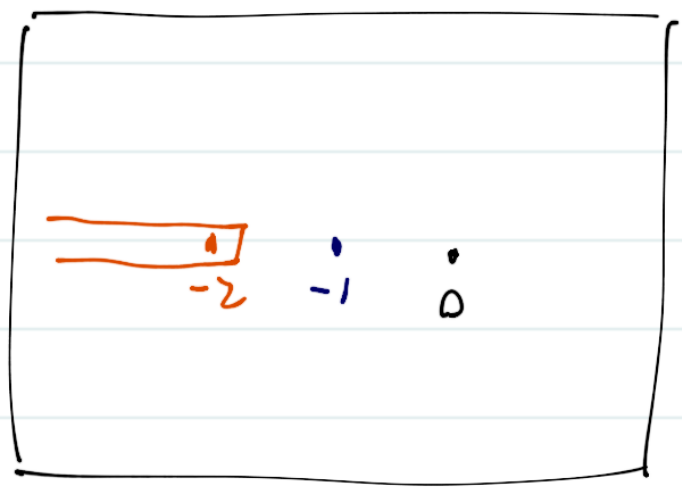
<u>Recall</u> Bases for $(U_0\mathfrak{sl}(2), N)$-mods

1) Simples: L_0, V_{-2}

2) <u>Verma</u>: V_0, V_{-2}
↓ (costandards)
easy to construct



3) coVerma: V_0', $V_{-2}' = V_{-2}$
(standards)

Space $\mathbb{C}[H]$

<u>Remark</u>: relating 1) $\leftrightarrow$ 2), 3) is Kazhdan-Lusztig theory

$$\begin{array}{c|cc} & \multicolumn{2}{c}{\text{simples}} \\ & 0 & -2 \\ \hline \text{Vermas}\ 0 & 1 & 1 \\ -2 & 0 & 1 \end{array}$$

4) <u>Projectives</u>

Projective covers

$$\begin{array}{cc} V_0 & T \\ \downarrow & \downarrow \\ L_0 & V_{-2} \end{array}$$

Indecomposible: L_0, V_{-2}, V_0, V_0' and T.

5) <u>Injectives</u>

$$\begin{array}{cc} V_0' & T \\ \uparrow & \uparrow \\ L_0 & V_{-2} \end{array}$$

Use projs to give quiver presentation

Proj generator. $V_0 \oplus T = P$

$$(\mathcal{U}_0\mathfrak{sl}(2), N)\text{-mod} \longrightarrow \mathrm{End}(P)\text{-mod}$$

$$M \longmapsto \mathrm{Hom}(P, M)$$

$$\begin{tikzcd}[column sep=huge] \underset{(V_0)}{\overset{\mathrm{Hom}(V_0, M)}{\bullet}} \arrow[r, bend right, "p^*"'] & \underset{(T)}{\overset{\mathrm{Hom}(T, M)}{\bullet}} \arrow[l, bend right, "q^*"'] \end{tikzcd}$$

$\mathrm{End}(V_0) = \mathbb{C}$

$\mathrm{End}(T) = \mathbb{C}[q^*, p^*]/(\varepsilon^2)$

$q^* p^* = 0$

$\varepsilon = p^* q^*, \quad \varepsilon^2 = 0$

$$\begin{tikzcd} & V_{-2} \arrow[dr] & \\ p : T \arrow[ur] \arrow[rr] & & V_0 \end{tikzcd}$$

$$q : V_0 \longrightarrow T$$

$$\begin{tikzcd} D_{\mathbb{A}^1}/(z\partial_z z) \arrow[rr] \arrow[dr] & & D_{\mathbb{A}^1}/(z\partial_z) \\ & D_{\mathbb{A}^1}/(z) \arrow[ur] & \end{tikzcd}$$

$$D_{\mathbb{A}^1}/(z\partial_z) \longrightarrow D_{\mathbb{A}^1}/(z\partial_z z)$$

Exer: Jordan-Holder series of T

$$0 \underset{\substack{\uparrow \\ V_{-2}}}{\subset} V_{-2} \underset{\substack{\uparrow \\ L_0}}{\subset} V_0 \underset{\substack{\uparrow \\ V_{-2}}}{\subset} T$$

simple quotient

Find quiver back in reps

$$XY = 0$$

$$(YX)^2 = 0$$

$$\begin{tikzcd} \bullet_{-6} & \bullet_{-4} & \bullet_{-2} \arrow[rr, bend right, "X"'] & \bullet_{-1} & \bullet_{0} \arrow[ll, bend right, "Y"'] \end{tikzcd}$$

use commutator

(quivers are reversed)
left dot is right dot...

Indecomposibles

1) L_0 $\qquad \begin{array}{ccc} 0 & \qquad & \mathbb{C} \\ \cdot & & \cdot \end{array}$

2) V_{-2} $\qquad \begin{array}{ccc} \mathbb{C} & \qquad & 0 \\ \cdot & & \cdot \end{array}$

3) V_0 $\qquad \begin{tikzcd} \mathbb{C} \arrow[r, bend right, "0"'] & \mathbb{C} \arrow[l, bend right, "\sim"'] \end{tikzcd}$

4) V_0' $\qquad \begin{tikzcd} \mathbb{C} \arrow[r, bend right, "\sim"'] & \mathbb{C} \arrow[l, bend right] \end{tikzcd}$

5) T $\qquad \begin{tikzcd} \mathbb{C} \oplus \mathbb{C} \arrow[r, bend left, "p_1"] & \mathbb{C} \arrow[l, bend left, "i_2"] \end{tikzcd}$

Find quiver geom. via Riemann-Hilbert corr.

Ideal: D-mods $\rightsquigarrow$ solutions (complexes of sheaves) of vector spaces

"We want derived solution" of PDE

.... derived category.

Concretely on curve X:

some D-mod.

$$M \longmapsto \left[\begin{array}{c} \Omega^1_X \otimes M \\ \uparrow \nabla \\ M \end{array}\right]^{0}_{-1}$$

analytic section

de Rham complex of M.

We'll obtain:

$$\begin{tikzcd}
(\mathcal{D}_{\mathbb{P}^1}, N)\text{-mods} \arrow[r, hook] \arrow[d, "\wr"] & D_N(\mathbb{P}^1) \\
\mathrm{Perv}_N(\mathbb{P}^1) \arrow[ur, phantom, "\subset"] &
\end{tikzcd}$$

we won't lose anything by looking at solutions →

derived cat. of N-eq-V sheaves on $\mathbb{P}^1$

Calc. for indecomposible

1) $$\begin{tikzcd}
L_0 \arrow[r, squiggly, "BB"] & \mathcal{O}_{\mathbb{P}^1} \arrow[r, squiggly, "RH"] & \Omega^1_{\mathbb{P}^1} & 0 & 0 \\
& & \mathcal{O}_{\mathbb{P}^1} \arrow[u, "d"] \arrow[r, phantom, "\simeq"] & \mathbb{C}_{\mathbb{P}^1}[1] & -1
\end{tikzcd}$$

2) $V_{-2} \longrightarrow \mathcal{D}_{\mathbb{A}^1}/(z) \rightsquigarrow$

$$\begin{array}{cc} \mathbb{C}_{\{0\}} & 0 \\ 0 & -1 \end{array}$$

$$\mathcal{O}_{\mathbb{A}^1\setminus\{0\}}/\mathcal{O}_{\mathbb{A}^1} \xrightarrow{\nabla} \mathcal{O}_{\mathbb{A}^1\setminus\{0\}}/\mathcal{O}_{\mathbb{A}^1} \otimes \Omega^1_{\mathbb{A}^1}$$

∂_z $\qquad\qquad$ dz

$$\cdots \frac{1}{z^3} \leftarrow \frac{1}{z^2} \xleftarrow{\partial_z} \frac{1}{z}$$

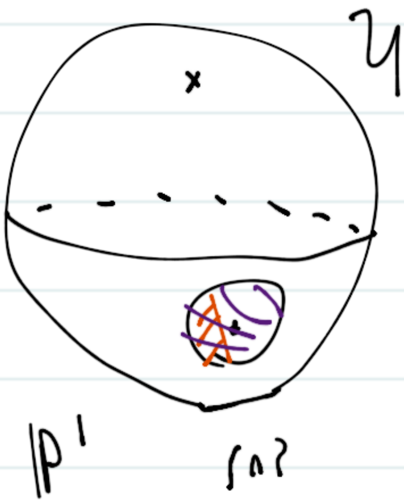
↑ trivialized 1-form by dz

Exer $V_0 \overset{BB}{\rightsquigarrow} \overset{RH}{\rightsquigarrow} j_!\mathbb{C}_U[1]$

$V_0' \rightsquigarrow \rightsquigarrow j_*\mathbb{C}_U[1]$ $\qquad U \overset{j}{\hookrightarrow} \mathbb{P}^1 \overset{i}{\hookleftarrow} \{0\}$

$T \rightsquigarrow \rightsquigarrow$ succesive extension $\qquad$ N-orbits

Finally quiver presentation geometrically



bdy in LES

relative sections of complex

stalk of complex (shift by 1)

$$\bullet \rightleftarrows \bullet$$

restriction

On the curve vanishing on the orange