Pick up our current goal: explain BB dictionary:

$$\begin{array}{ccc} U_0\, sl(2)\text{-module} & \underset{\Gamma}{\overset{L \text{ localization functor}}{\rightleftarrows}} & D_{\mathbb{P}^1}\text{-mods} \\ \text{trivial rep: } L_0 & \longleftrightarrow & \mathcal{O}_{\mathbb{P}^1} \\ U_0\, sl(2) & \longleftrightarrow & D_{\mathbb{P}^1} \end{array}$$

<u>Understand:</u> highest weight module $\longleftrightarrow$?

<u>Recall</u> $K \subset SL(2)$ subgrp.

$(U sl(2), K)$-mod is a $U sl(2)$-mod and K-rep. M st:

1) $U sl(2) \otimes M \to M$ is a K-rep map.
 ↑ also a k-rep by adj. action

2) differentiation of K-rep gives $U sl(2)$-mod restricted to Uk.

for "weak" $(U sl(2), K)$-mod, only (1) holds.

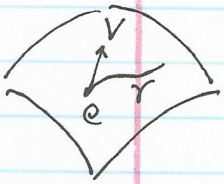
<u>Ex</u>. V_λ Verma modules are $(U sl(2), N)$-mods.

$$sl(2): \quad H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad n = \langle X \rangle,\ b = \langle H, X \rangle \qquad N = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\},\ B = \left\{ \begin{pmatrix} z & u \\ 0 & z^{-1} \end{pmatrix} \right\}$$

<u>Def</u>. $K \circlearrowright X$ smooth variety;
↑ <u>affine grp</u>, (D_X, K)-mod $:=$ (or strongly K-equivariant D_X-mod) is a D_X-mod and <u>K-equivariant</u> quasicoherent sheaf $\mathcal{M}$.

st: 1) $D_X \otimes \mathcal{M} \to \mathcal{M}$ is K-equivariant (translate everything inside)

2) differentiation of K-equivariant gives D_X-action restricted to Uk.



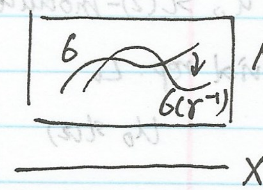
K.

$\dot{\gamma}_0 = v$

2 ways to differentiate sections of M:

1) Using K-equivariant. $\sigma \in \Gamma(M)$

$$(\nabla^1_v \sigma)(x) = \lim_{h\to 0} \frac{\sigma(\gamma_h^{-1}x) - \sigma(x)}{h}$$



using equivariant

2) Using D_X-structure $\quad K \curvearrowright X \rightsquigarrow k \to Vect(X)$

$$\nabla^2_v \sigma = \nabla_v \sigma \leftarrow D_X\text{-mod. str.}$$

2) in def says that $\nabla^1 = \nabla^2$ on k.

again, "weak" means 1) only.

Exer. BB equiv. match weakly/strongly K-equiv. objects.

Conclusion: Verma modules localize to $(D_{\mathbb{P}^1}, N)$-mods.

N-orbits on $\mathbb{P}^1$ (Schubert cells)

2 coord. patches ($\mathbb{A}^1$'s)

$$\begin{cases} X \mapsto -z^2\partial_z \\ H \mapsto 2z\partial_z \\ Y \mapsto \partial_z \end{cases}$$

$$\begin{cases} X \mapsto \partial_w \\ H \mapsto -2w\partial_w \\ Y \mapsto -w^2\partial_w \end{cases}$$

Here $N \curvearrowright \mathbb{A}^1$ by translation simply transitive

Exer. Cat. of (D_X, K)-mod where K acts simply transitively is equiv. to $Vect : \mathcal{O}_X \mapsto \mathbb{C}$

sum of copies

$\Downarrow$ 2nd chart is trivial.

analog: open affine space in some flag var.

Conclusion: Cat of $(D_{\mathbb{P}^1}, N)$-mods is equiv. to cat of $D_{\mathbb{A}^1}$-mods where restriction away from $0 \in \mathbb{A}^1$ is trivialized (isom. to $\oplus$ of copies of $\mathcal{O}$)

Let's construct some modules over Weyl alg. $D_{\mathbb{A}^1}(\mathbb{A}^1) = \langle z, \partial z \rangle$ ← 1st chart.

↑ much easier alg. than enveloping alg.!

(POV: "D-modus are linear algebraic PDEs")

Ex. $P(f) = 0$. linear alg. PDE $\rightsquigarrow$ $M = D_X / D_X(P)$ D_X-mod) no need to solve, just encode info.

1) $M = D_{\mathbb{A}^1} / D_{\mathbb{A}^1}(\partial_z) \cong \mathcal{O}_{\mathbb{A}^1}$ $\boxed{\text{func}}$, restricted away from 0: trivial / (algebra auto.) F.T. $z \longleftrightarrow \partial z$ ...

$1 \mapsto 1$ ← q.s. sheaf supported everywhere

2) $M = D_{\mathbb{A}^1} / D_{\mathbb{A}^1}(z) \cong \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}} / \mathcal{O}_{\mathbb{A}^1}$: $\boxed{\delta\text{-func}}$ at 0

$1 \mapsto \frac{1}{z}$ "residue" ← q.s. sheaf

Under Γ: 1) $\rightsquigarrow L_0$

2) sections: $\cdots \frac{1}{z^3} \xrightarrow{X} \frac{1}{z^2} \xrightarrow{X} \frac{1}{z} \xrightarrow{X = -z^2 \partial_z} 0$

$H \circlearrowleft$ Y $\circlearrowleft_H$ Y $\circlearrowleft$ $H = 2z\partial_z$

weights: -6 -4 -2 ← heighest weight

V_{-2}

L_0, V_{-2}: 2 simple (U_0, N)-mods

Let's find BGG resolution: $V_{-2} \hookrightarrow V_0 \twoheadrightarrow L_0$

$D_{\mathbb{A}^1} / D_{\mathbb{A}^1}(z) \hookrightarrow D_{\mathbb{A}^1} / D_{\mathbb{A}^1}(z\partial_z) \twoheadrightarrow D_{\mathbb{A}^1} / D_{\mathbb{A}^1}(\partial_z)$, all cyclicly generated

$1 \mapsto \partial_z$ $\qquad$ $1 \mapsto 1$

• only Verma modules w/ central character L_0 are V_0, V_{-2}

• $\cancel{V_{-4}}$

Under Γ:

$$\cdots\ \partial_z^3 \underset{Y}{\overset{X}{\rightleftarrows}} \partial_z^2 \underset{Y}{\overset{X}{\rightleftarrows}} \partial_z \underset{Y}{\overset{X}{\rightleftarrows}} 1 \xrightarrow{X} 0$$

$$\begin{array}{cccc} -6 & -4 & -2 & 0 \end{array}$$

after shuffling symbols in Weyl alg

Let's find all indecomposible $(U_0 sl(2), N)$-mods

so far, L_0, V_{-2} simple.

V_0 indecomposible

Two others : (1) $M = D_{A^1}/D_{A^1}(\partial_z z)$

(as a D-module theorists)

$$\cdots\ \partial_z^2 \rightleftarrows \partial_z \rightleftarrows 1 \rightleftarrows z$$

$$\begin{array}{cccc} -6 & -4 & -2 & 0 \end{array}$$

$Y = \partial_z$

cofree

$L_0 \hookrightarrow V_0' \to V_{-2}$

co-Verma module

Exer $V_2' = V_2$.

"Bases" of $(U_0 sl(2), N)$-modes:

simples: V_{-2}, L_0

Vermas : V_0, V_{-2}

co-Vermas: V_0', $V_{-2}' = V_{-2}$

projective: every simple has a proj. cover

$$V_0 \downarrow L_0, \quad T = D_{A^1}/D_{A^1}(z\partial_z z) \downarrow V_{-2}, \ \ldots$$

requirement: restriction away from 0 is sum of copies of $\mathcal{O}$.

Exer Jordan-Holder thm for this module