Lec 06

Current goal : Explore BB dictionary

$$\mathcal{U}_0\, sl(2)\text{-mods} \underset{\Gamma}{\overset{L}{\rightleftarrows}} \mathcal{D}_{\mathbb{P}^1}\text{-modules}$$

$$L_0 \longleftrightarrow \mathcal{O}_{\mathbb{P}^1}$$

$$\mathcal{U}_0\, sl(2) \longleftrightarrow \mathcal{D}_{\mathbb{P}^1}$$

Recall : Let $K \subseteq SL(2)$ be a subgroup
$(\mathcal{U}\, sl(2), K)$ - mod is a $\mathcal{U}\, sl(2)$ - module and a K-rep M s.t

1) $\mathcal{U}\, sl(2) \otimes M \longrightarrow M$ is a K-rep. map
2) Differentiation of K-rep. gives $\mathcal{U}\, sl(2)$-mod restricted to $\mathcal{U}k$.

Ex V_λ, Verma mods are $(\mathcal{U}\, sl(2), N)$- mods

$$\left[\; sl(2) \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right.$$

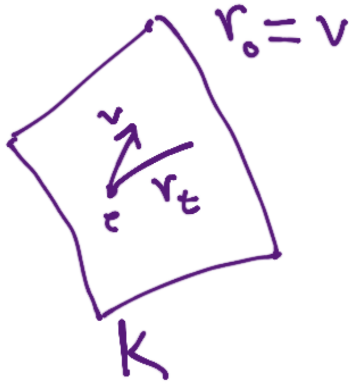
$$\mathfrak{n} = \langle X \rangle \qquad \mathfrak{b} = \langle H, X \rangle$$

$$N = \left\langle \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} z & u \\ 0 & z^{-1} \end{pmatrix} \right\rangle$$

Defn Let K be an affine group acting on a smooth variety X.
Then (D_X, K)-mod (or strongly K-equivariant D_X-mod) is a D_X-mod and K-equivariant qc sheaf $\mathcal{M}$.

1) $D_X \otimes \mathcal{M} \to \mathcal{M}$ is K-equivariant

2) Differentiation of the K-equivariance gives a D_X-action restricted to $\mathcal{U}\mathfrak{k}$



Two ways to differentiate sections of $\mathcal{M}$:

1) Using K-equiv. $\sigma \in \Gamma(\mathcal{M})$ we consider

$$\nabla^1_v \sigma := \lim_{h \to 0} \frac{\sigma(r_h^{-1} x) - \sigma(x)}{h}$$

2) Using D_X-structure: $K \curvearrowright X \rightsquigarrow K \rightsquigarrow \mathrm{Vect}(X)$

$$\nabla^2_v \sigma = \nabla_v \sigma \leftarrow D_X\text{-mod str.}$$

Now this just says $\nabla^1 = \nabla^2$.

Exercise) BB-equivs match weakly/strongly K-equiv. objects.

Conclusion: Verma modules localize to $(D_{\mathbb{P}^1}, N)$-modules

N-orbits on $\mathbb{P}^1$ (Schubert cells)

Two coordinate $\mathbb{A}^1$'s: $\begin{cases} X \longmapsto -z^2 \partial_z \\ H \longmapsto 2z\partial_z \\ Y \longmapsto \partial_z \end{cases}$

$\begin{cases} X \longmapsto \partial_w \\ H \longmapsto -2w\partial_w \\ Y \longmapsto -w^2\partial_w \end{cases}$ Here $N \curvearrowright \mathbb{A}^1$ by translation. It's simply transitive

Exercise) category of (D_X, K)-mod where K acts simply transitively is equivalent to Vect $\mathcal{O}_X \mapsto \mathbb{C}$

Conclusion: Category of $(D_{\mathbb{P}^1}, N)$-mods is equivalent to the category of $D_{\mathbb{A}^1}$-mods whose restriction away from $0 \in \mathbb{A}^1$ is trivialized (isom. to $\oplus$ copies of $\mathcal{O}$)

Let's construct some modules over the Weyl algebra $D(\mathbb{A}^1) = \langle z, \partial_z \rangle$

(POV: "D-mods are lin. alg. PDEs"

Ex $P(f) = 0$ lin alg PDE $\rightsquigarrow \mathcal{M} = D_X / D_X(P)$ D_X-mod)

1) $m = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}(\partial_z) \simeq \mathcal{O}_{\mathbb{A}^1}$

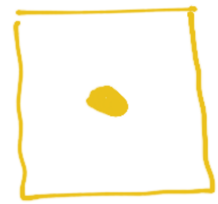
$$1 \longmapsto 1$$

functions

2) $m = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}(z) \simeq \mathcal{O}_{\mathbb{A}^1 \setminus \{0\}} / \mathcal{O}_{\mathbb{A}^1}$

$$1 \longmapsto \frac{1}{z}$$

Delta functions at 0



Under Γ: 1) $\rightsquigarrow L_0$

2) $\rightsquigarrow V_{-2}$

$$\begin{array}{cccc} & & & \times \\ \cdots\cdots & \frac{1}{z^3} & \frac{1}{z^2} & \frac{1}{z} \\ & & & Y \quad H \\ & -6 & -4 & -2 \end{array}$$

Fact: These are the only simple $(\mathcal{U}_0, N)$-modules.

Let's find BGG resolutions $V_{-2} \hookrightarrow V_0 \twoheadrightarrow L_0$

$$\mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}(z) \hookrightarrow \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}(z\partial_z) \twoheadrightarrow \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}(\partial_z)$$

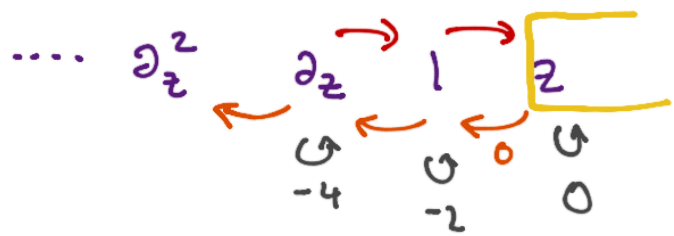
Under Γ:

$$\begin{array}{ccccc} \cdots\cdots & \partial_z^3 & \partial_z^2 & \partial_z & \xrightarrow{0} 1 \\ & -6 & -4 & -2 & 0 \end{array}$$

Let's find all indecomposable $(\mathcal{U}_0 sl(2), N)$-mods.
So far we have L_0, V_{-2} as the simple ones and V_0 an indecomposable.

We have 2 others:

1) $M = D_{A'} / D_{A'}(\partial_z z)$



$$L_0 \hookrightarrow V_0' \to V_{-2}$$

coVerma

One can check that $V_2' = V_2$

"Bases" of $(\mathcal{U}_0 sl(2), N)$-mods

Simples L_0, V_{-2}

Vermas V_0, V_{-2}

coVermas $V_0', V'_{-2} = V_{-2}$

Projectives $\begin{array}{c} V_0 \\ \downarrow \\ L_0 \end{array}$, $\begin{array}{c} T \\ \downarrow \\ V_{-2} \end{array} = D_{A'} / D_{A'}(z \partial_z z)$