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?

Beilinson–Bernstein for su(2)

Recall Classical vs Quantum picture

Poisson algebra →

$$\begin{array}{ccc} \mathfrak{sl}(2) \overset{*}{\cong} \mathfrak{sl}(2) & \hookleftarrow & N \\ \big\downarrow \text{det/Killing form} & & \big\downarrow \\ \mathbb{A}^1 & \hookleftarrow & \{0\} \end{array}$$

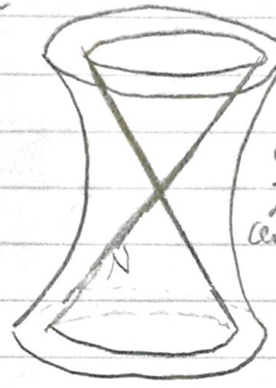
Killing form

nilpotent cone

central

Classical

fibres are symplectic leaves



$$\begin{array}{ccc} U\mathfrak{sl}(2) & \twoheadrightarrow & \mathcal{U}_0 = U(\mathfrak{sl}(2))/\mathfrak{z}_0 \\ \big\uparrow \text{Casimir} & & \\ \mathfrak{z} = \mathbb{C}[C] & \hookleftarrow & \mathfrak{z}_0 \end{array}$$

central

quantum

z-ideal annihilating trivial rep. L_0

Blowing up N at $\{0\}$ gives a resolution $T^*\mathbb{P}^1$

Exer $T^*\mathbb{P}^1 = \{ l \in \mathbb{P}^1, A \in \mathfrak{sl}(2) \mid Al = 0 \}$

Note: $\mathcal{O}(N) \xrightarrow[\pi^*]{\sim} \mathcal{O}(T^*\mathbb{P}^1)$

BB localiz. for $\mathfrak{sl}(2)$ with trivial central character

Fundamental fact 1: $\mathcal{U}_0 \xrightarrow{\sim} \Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})$

We'll discuss the detailed proof later

Fundamental fact 2: "$\mathbb{P}^1$ is D-affine"

(Def For X smooth, a D_X-module M is a quasicoherent sheaf on X with a compatible D_X-action)

$$\Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})\text{-mod} \underset{\Gamma}{\overset{L}{\rightleftarrows}} D_{\mathbb{P}^1}\text{-mod} \qquad L \dashv \Gamma$$

localization, global sections

inverse equivalences

Proof later

Combining 1+2 $\rightsquigarrow$ $\mathcal{U}_0\text{-mod} \cong D_{\mathbb{P}^1}\text{-mod}$

$$L(M) = D_{\mathbb{P}^1} \underset{\Gamma(\mathbb{P}^1, D_{\mathbb{P}^1})}{\otimes} M$$

Dictionary

	$\mathcal{U}_0\, sl(2)$	$D_{\mathbb{P}^1}$
	$\mathcal{U}_0\, sl(2)$	$D_{\mathbb{P}^1}$
triv	L_0	$\mathcal{O}_{\mathbb{P}^1}$
Verma mod	V_0	
$V_{-2} \hookrightarrow V_0 \twoheadrightarrow L_0$	V_{-2}	$\hookrightarrow \quad \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}$

Recall $V_\lambda = \mathcal{U}sl(2) \otimes_{\mathcal{U}b} \mathbb{C}_\lambda$

$SL(2) \supseteq B \to b = \mathbb{C}\langle H, X\rangle$ upper triangular

$N \to n = \mathbb{C}\langle X\rangle$ strictly upper triangular

$H \to h = b/n$

<u>Obs</u> $N \circlearrowright V_\lambda$, ie. X acts "locally finitely" on V_λ so $\exp(X)$ mal
s

Def $K \leq SL(2)$ an algebraic group with $k = Lie(K) \leq sl(2)$

A $(\mathcal{U}sl(2), K)$-mod M is a $\mathcal{U}sl(2)$-mod and a K-rep such that

① $\mathcal{U}sl(2) \otimes M \to M$ is K-equivariant

② Differential of $K \circlearrowright M$ agrees with the $\mathcal{U}k$-restriction of $\mathcal{U}sl(2)$ action

A <u>weak</u> $(\mathcal{U}sl(2), K)$-module doesn't necessarily satisfy ②

<u>Ex</u> 1) L_n are $(\mathcal{U}sl(2), SL(2))$-mods $= SL(2)$-reps

2) V_λ are $(\mathcal{U}sl(2), N)$-mods

<u>Next time</u> Enhance dictionary to

$$(\mathcal{U}_0, K)\text{-mod} \xrightarrow{\sim} (D_{\mathbb{P}^1}, K)\text{-mod}.$$

We will use this to find $D_{\mathbb{P}^1}$-mod corresponding to K